AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 6 Notes: Ratio & Root Tests

The RATIO TEST and the ROOT TEST are tests for absolute convergence.

The RATIO TEST works well for series with exponentials and factorials.

> **RATIO TEST**
>
> Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.
>
> 1. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
> 2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
> 3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive.

Examples: Use the Ratio Test to determine convergence or divergence of each series.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$a_n = \frac{2^n}{n!}$ $\quad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$\lim_{n\to\infty}\left|\frac{\cancel{2^{n}}\,2}{(n+1)(\cancel{n!})}\cdot\frac{\cancel{n!}}{\cancel{2^n}}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = \frac{2}{\infty} = 0 < 1$

$\boxed{\text{Converges absolutely by Ratio Test}}$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)!}{2^{4n}}$

$a_n = \frac{(-1)^n(2n+1)!}{2^{4n}}$ $\quad a_{n+1} = \frac{(-1)^{n+1}(2(n+1)+1)!}{2^{4(n+1)}}$

$\lim_{n\to\infty}\left|\frac{(2n+3)(2n+2)\cancel{(2n+1)!}}{2^{\cancel{4n}+4}}\cdot\frac{\cancel{2^{4n}}}{\cancel{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(2n+3)(2n+2)}{2^4}\right| = \infty$

$\boxed{\text{Diverges by Ratio Test}}$

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$a_n = \frac{1}{\sqrt{n}}$ $\quad a_{n+1} = \frac{1}{\sqrt{n+1}}$

$\lim_{n\to\infty}\left|\frac{1}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{1}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \sqrt{1} = 1$

Ratio Test is inconclusive

$\boxed{\text{p-series test}}$ $p = 1/2$, $\boxed{\text{diverges}}$

The Root Test works great for series with nth powers.

ROOT TEST

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Examples: Determine the convergence or divergence of each series with the Root Test.

1. $\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{\pi}{4}\right)^n\right|} = \lim_{n\to\infty} \left(\frac{\pi}{4}\right) = \frac{\pi}{4} < 1$$

$\boxed{\text{Converges by Root Test}}$

2. $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{-3n}{2n+1}\right|^{3n}} = \lim_{n\to\infty} \left(\frac{3n}{2n+1}\right)^3$$

$$= \left(\frac{3}{2}\right)^3 = \frac{27}{8} > 1$$

$\boxed{\text{Diverges by Root Test}}$