AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 5 Notes: Alternating Series

An alternating series has terms that alternate between positive and negative:

$$\sum_{n=1}^{\infty}(-1)^n a_n \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

For example, this is a common alternating series:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^n}{n}+\cdots$$

ALTERNATING SERIES TEST

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if both of these conditions are met:

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all n (each term must be $\le$ the preceding term).

Examples: Determine convergence or divergence.

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

$a_n = \frac{1}{n}$

$\lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{\infty} = 0$ ✓

$\frac{1}{n+1} \le \frac{1}{n}$ ✓

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by Alternating Series Test

2. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^2+1}$

$a_n = \frac{n^2}{n^2+1}$

$\lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right) = 1 \neq 0$

Diverges by n^{th} term test

REMAINDER THEOREM FOR ALTERNATING SERIES

If a convergent alternating series has R_N as the remainder obtained by approximating the sum of the series S with S_N, then

$$\boxed{|R_N| \le a_{n+1}}$$

***What this really means: The remainder after the nth partial sum S_N is always less than or equal to the first omitted term of the alternating series.

Examples:

1. Find the number of terms needed to approximate $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ with an error less than 0.001.

$$a_n = \frac{1}{2^n n!}$$

Start with $|R_N| \le a_{n+1} \le 0.001$.

$$a_{n+1} = \frac{1}{2^{n+1}(n+1)!} \le 0.001$$

plug in values of n.

$$\frac{1}{2^{3+1}(3+1)!} = .003 \not\le .001 \qquad \frac{1}{2^{4+1}(4+1)!} = \frac{1}{3840} = .0002 \le .001 \checkmark$$

$n=4$

$\boxed{\text{5 terms}}$ since starts at $n=0$

2. Find the number of terms needed to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ with an error less than 0.001.

$$a_n = \frac{1}{n^4}$$

$$a_{n+1} = \frac{1}{(n+1)^4} \le .001$$

$$\frac{1}{(3+1)^4} = .004 \not\le .001 \qquad \frac{1}{(4+1)^4} = .0016 \not\le .001$$

$$\frac{1}{(5+1)^4} = .0007 \le .001 \checkmark$$

$n=5$

$\boxed{\text{5 terms}}$

ABSOLUTE CONVERGENCE OF AN ALTERNATING SERIES

Let $\sum_{n=1}^{\infty}(-1)^n a_n$ be an alternating series.

1. $\sum_{n=1}^{\infty}(-1)^n a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} a_n$ converges.

2. $\sum_{n=1}^{\infty}(-1)^n a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty}(-1)^n a_n$ converges

   but $\sum_{n=1}^{\infty} a_n$ diverges.

Examples: Does each series converge or diverge? If it converges, is it absolutely or conditionally convergent?

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ $\quad a_n = \frac{1}{n+1}$ $\quad \lim_{n\to\infty}\left(\frac{1}{n+1}\right) = \frac{1}{\infty} = 0$ ✓

① $\frac{1}{(n+1)+1} \leq \frac{1}{n+1}$ ✓

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ converges by Alt. Series Test

② $\sum_{n=1}^{\infty}\frac{1}{n+1}$ $\quad a_n = \frac{1}{n+1}$ $\quad b_n = \frac{1}{n}$ bigger

$\sum_{n=1}^{\infty}\frac{1}{n} \to$ p-series test, $p=1$, diverges

$\boxed{\text{Conditionally convergent}}$

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sqrt{n}}$ ① $a_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty}\frac{1}{n^{3/2}} = \frac{1}{\infty} = 0$ ✓ $\quad \frac{1}{(n+1)^{3/2}} \leq \frac{1}{n^{3/2}}$ ✓

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sqrt{n}}$ converges by Alt. Series Test

② $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \leftarrow$ p-series test, $p=3/2$, $p>1$ converges

$\boxed{\text{absolutely convergent}}$