AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 4 Notes: Direct & Limit Comparison Tests

DIRECT COMPARISON TEST Let $0 \le a_b \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

In other words.....

1. If the larger series converges, the smaller must also converge. However, if the larger series diverges, we can't tell about the smaller one. Try a different test!

2. If the smaller series diverges, so will the larger one.

3. *Remember, when choosing a series with which to compare the given series, choose one with the same magnitude or degree*

Examples:

$a_n < b_n$ $\quad \frac{1}{5} < \frac{1}{3}$

1. Does $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converge or diverge. $\quad a_n = \frac{1}{3n^2+2} \quad b_n = \frac{1}{3n^2}$ bigger

$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ p-series test, $p > 1$, converges

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{3n^2+2} \text{ converges by Direct Comparison Test}}$

2. Show that $\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges. $\quad a_n = \frac{1}{n-1}$ bigger $\quad b_n = \frac{1}{n}$ $\quad \frac{1}{3} > \frac{1}{4}$ $\quad a_n > b_n$

$\sum_{n=2}^{\infty} \frac{1}{n}$, p-series Test, $p = 1$, diverges

$\boxed{\therefore \sum_{n=2}^{\infty} \frac{1}{n-1} \text{ diverges by Direct Comparison Test}}$

3. Does $\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$ converge or diverge? $\quad a_n = \frac{2^n}{3^n+5} \quad b_n = \frac{2^n}{3^n}$ bigger

$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \rightarrow$ Geometric series test, $r < 1$, converges

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{2^n}{3^n+5} \text{ converges by Direct Comparison Test}}$

LIMIT COMPARISON TEST

Suppose $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$, where L is finite and positive. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either **both converge** or **both diverge**.

Remember, when choosing a series with which to compare the given series, choose one with the same magnitude or degree.

Examples: Test for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

$a_n = \frac{1}{3n^2-4n+5}$ $\quad b_n = \frac{1}{3n^2}$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{3n^2-4n+5}\cdot\frac{3n^2}{1}\right) = \lim_{n\to\infty}\left(\frac{3n^2}{3n^2-4n+5}\right) = 1 = L \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}$, p-series test, $p=2$, converges

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5} \text{ converges by Limit Comparison Test}}$

$\frac{n^3}{n^4} = \frac{1}{n}$

2. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$

$a_n = \frac{n^{k-1}}{n^k+1}$ $\quad b_n = \frac{n^{k-1}}{n^k} = \frac{1}{n}$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n^{k-1}}}{n^k+1}\cdot\frac{n^k}{\cancel{n^{k-1}}}\right) = \lim_{n\to\infty}\frac{n^k}{n^k+1} = 1 = L \checkmark$$

$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k} = \sum_{n=1}^{\infty}\frac{1}{n}$, p-series test, $p=1$, diverges

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1} \text{ diverges by Limit Comparison Test}}$