AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 3 Notes: Integral & p-Series Test

The Integral Test

If f is positive, continuous, and decreasing for $x > 1$ and $a_n = f(n)$,

then $\boxed{\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx}$ either both converge or both diverge.

$a_1 \quad a_2 \quad a_3 \quad a_4$

Example 1: Does the series $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$ converge or diverge?

First write the series in summation notation:

$2(1)+1=3$

$2(2)+1=5$

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{2n+1} \text{ diverges by Integral Test}}$

Now apply the integral test:

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{2}{2x+1}\,dx \qquad u=2x+1 \quad du=2dx$$

$$\frac{1}{2}\int \frac{du}{u}$$

$$\frac{1}{2}\ln|2x+1|\Big]_1^b = \frac{1}{2}\ln|2b+1| - \frac{1}{2}\ln|2(1)+1|$$

diverges

$$\lim_{b\to\infty}\left(\frac{1}{2}\ln|2b+1| - \frac{1}{2}\ln|3|\right) = \infty - \frac{1}{2}\ln 3 = \infty$$

Example 2: Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx$$

$$\lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx \qquad u=x \quad du=dx \quad a=1$$

$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges by Integral Test}}$

$$\int \frac{du}{u^2+a^2}$$

$$\arctan(x)\Big]_1^b = \arctan b - \arctan 1$$

$$\lim_{b\to\infty}(\arctan b - \pi/4) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

converges

Note: In example 2, the integral converges to $\pi/4$. This does NOT mean that $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \pi/4$. It just means that the series converges.

p-Series and Harmonic Series

A series in the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is called a **p-series**.

When p = 1, we have the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Convergence of p-Series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

Converges if $p > 1$

Diverges if $0 < p \le 1$

Example 4:

Does $\sum_{n=1}^{\infty} \frac{n^5}{n^7}$ converge or diverge?

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $p=2$ $\boxed{\text{converges}}$ by p-series test since $p>1$

Your Turn #1:
Use the Integral Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

$\int_1^{\infty} \frac{1}{x+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+1}$ $u=x+1$, $du=dx$

$\int \frac{du}{u}$

$\ln|x+1| \Big]_1^b$

$\ln|b+1| - \ln|1+1|$

$\lim_{b\to\infty} \left(\ln|b+1| - \ln 2\right)$

$= \infty - \ln 2 = \infty$

diverges

$\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges

Your Turn #2:
Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$ $p = 1/5$

$\boxed{\text{diverges}}$ by p-series test since $0 < p \le 1$