AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 2 Notes: Series & Convergence

Definition of Infinite Series:
If $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series. Sum!

Partial Sums
Consider the sequence of partial sums $S_1, S_2, S_3, \ldots, S_n, \ldots$
Where $S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

If a sequence of partial sums converges, then the infinite series converges.

$$\lim_{n\to\infty}\{S_n\} = S \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Example 1: Find the first four terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$a_1: \frac{1}{1^2} = 1$
$a_2: \frac{1}{2^2} = \frac{1}{4}$
$a_3: \frac{1}{3^2} = \frac{1}{9}$
$a_4: \frac{1}{4^2} = \frac{1}{16}$

$S_1 = 1$
$S_2 = 1 + \frac{1}{4} = \frac{5}{4}$
$S_3 = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$
$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{205}{144}$

Geometric Series

$$\sum_{n=0}^{\infty} ar^n, a \neq 0$$

a = 1st term
r = ratio

each # in sequence is mult. by r to get to next #

1) A geometric infinite series **converges** if $|r| < 1$. It converges to $S = \frac{a}{1-r}$
2) A geometric infinite series **diverges** if $|r| \geq 1$.

geometric

Example 2:
Find the sum of the series:

$$a\ 2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\cdots$$

$\frac{\frac{3}{2}}{2}=\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}$ $r=3/4$ (each term mult. by $3/4$)

converge b/c $r<1$

$$S=\frac{a}{1-r}$$

$$=\frac{2}{1-\frac{3}{4}}=\frac{2}{\frac{1}{4}}=2\cdot 4=\boxed{8}$$

Example 3:
Write the decimal $0.23\overline{23}$ as a geometric series and write its sum as the ratio of two integers.

$0.23+0.0023+0.000023+\ldots$

$a=0.23$

$r=\frac{.0023}{.23}=0.01$

$$\boxed{\sum_{n=1}^{\infty}0.23(.01)^n}$$

Converges b/c $r<1$

$$S=\frac{0.23}{1-0.01}=\frac{.23\,(100)}{.99\,(100)}=\boxed{\frac{23}{99}}$$

Telescoping Series:

$$\sum_{n=1}^{\infty} b_n - b_{n-1}$$ } can often be obtained using partial fractions

Example 4:
Let $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$
Find the **sum** of the telescoping series.

$$=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-\frac{1}{\infty}=1-0=1$$

$\boxed{\text{Converges}}$ $\boxed{\text{sum}=1}$

Example 5: Write the series below in telescoping form and find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$\frac{1}{n(n+2)} = \frac{A^{(n+2)}}{n_{(n+2)}} + \frac{B^{(n)}}{n+2_{(n)}}$

$1 = A(n+2) + B(n)$

$\underline{n=-2}: \ 1 = A(\cancel{-2+2}) + B(-2)$

$B = -1/2$

$\underline{n=0}: \ 1 = A(0+2) + \cancel{B(0)}$

$A = 1/2$

$$\sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2} = \boxed{\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}}$$

$$\frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)\right.$$

$$\frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+2}\right]$$

$$\lim_{n\to\infty} \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{\infty} - \frac{1}{\infty}\right] = \frac{1}{2}\left(\frac{3}{2}\right) = \boxed{\frac{3}{4}}$$

nth Term Test for Divergence:

If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum a_n$ *diverges.*

Determine whether each series converges or diverges.

Example 6:

$$\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

$\boxed{\text{diverges}}$ by nth term test

Example 7:

$$\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n$$

geometric series

$r < 1$ $\boxed{\text{converge}}$

$$S = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 3 \cdot \frac{3}{1} = \boxed{9}$$

Example 8:

Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

$\boxed{\text{Converges}}$

$\frac{1}{n(n+3)} = \frac{A^{(n+3)}}{n_{(n+3)}} + \frac{B^{(n)}}{n+3_{(n)}}$

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

$1 = A(n+3) + B(n)$

$$= \frac{1}{3}\left[\left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + \left(\frac{1}{n} - \frac{1}{n+3}\right)\right]$$

$\underline{n=-3}: \ 1 = A(\cancel{-3+3}) + B(-3)$

$B = -1/3$

$\underline{n=0}: \ 1 = A(0+3) + \cancel{B(0)}$

$A = 1/3$

$$\lim_{n\to\infty} \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3} - 0 - 0 - 0\right]$$

$$= \frac{1}{3}\left[\frac{11}{6}\right] = \boxed{\frac{11}{18}}$$