AP Calculus BC
Unit 9 – Sequences & Series (Part 1)

Day 1 Notes: Sequences

A **sequence** is a list of terms: $a_1, a_2, a_3, \ldots, a_n, \ldots$
a_1 = first term
a_n = nth term
$\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$

Limit of a Sequence:

Let f be a function of real variables such that $\lim_{x\to\infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$.

Example 1:
Write out the first four terms of the sequence and then find the limit of the sequence with the nth term:

$$a_n = \frac{\ln n^2}{n}$$

$a_1 = \frac{\ln(1)^2}{1} = \ln 1 = 0$

$a_2 = \frac{\ln(2)^2}{2} = \frac{\ln 4}{2}$

$a_3 = \frac{\ln(3)^2}{3} = \frac{\ln 9}{2}$

$a_4 = \frac{\ln(4)^2}{4} = \frac{\ln 16}{4}$

$0, \frac{\ln 4}{2}, \frac{\ln 9}{2}, \frac{\ln 16}{4}, \ldots$

$\lim_{n\to\infty} \frac{\ln(n^2)}{n}$

Use L'Hopital's Rule

$= \lim_{n\to\infty} \frac{\frac{2n}{n^2}}{1}$

$= \lim_{n\to\infty} \frac{2}{n} = \frac{2}{\infty} = \boxed{0}$

Converges

Example 2:
Find the limit of $\{a_n\} = \frac{(n+1)!}{n!}$

$\lim_{n\to\infty} \frac{(n+1)!}{n!}$

$= \lim_{n\to\infty} \frac{(n+1)(\cancel{n!})}{\cancel{n!}}$

$= \lim_{n\to\infty} n+1 = \infty + 1$

$\boxed{\text{diverges}}$

Example 3:
Find the limit of $\{a_n\} = \{3 + (-1)^n\}$

$\lim_{n\to\infty} (3 + (-1)^n)$

Write out 1st few terms

$a_1 = 3+(-1)^1 = 2$
$a_2 = 3+(-1)^2 = 4$
$a_3 = 3+(-1)^3 = 2$
$a_4 = 3+(-1)^4 = 4$

goes back & forth b/t 2 & 4

$\boxed{\text{diverges}}$

Example 4:
Find the limit of $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$

$\lim_{n\to\infty} \frac{n}{1-2n}$

Remember you can look at horizontal asymptote. (leading coeff if same degree)

$= \boxed{-\frac{1}{2}}$

Monotonic Sequence:

$\{a_n\}$ is monotonic if

$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le \ldots$ (increasing) OR

$a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge \ldots$ (decreasing)

Bounded Sequence:

1) $\{a_n\}$ is bounded above if $a_n \le M$.
2) $\{a_n\}$ is bounded below if $a_n \ge N$.
3) $\{a_n\}$ is bounded if $N \le a_n \le M$ ← above & below

If a sequence is bounded and monotonic, then it will converge.

Example 5:

Show that a_n is bounded and monotonic.

$$\{a_n\} = \left\{\frac{1}{n}\right\}$$

Bounded:

$a_1 = \frac{1}{1} = 1$ ← 1st term

$\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$ ← nth term

converges to 0

$\boxed{\text{Bounded by } 0 \le a_n \le 1}$

Monotonic: $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots$

$\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2} > \frac{1}{n+3}$

decreasing

$\boxed{\text{monotonic}}$

Example 6:

Show that a_n is monotonic and bounded below.

$$\{a_n\} = \left\{\frac{n^2}{n+1}\right\}$$

Bounded:

$a_1 = \frac{1^2}{1+1} = \frac{1}{2}$

$\lim_{n\to\infty} \left(\frac{n^2}{n+1}\right) = \lim_{n\to\infty} \frac{2n}{1} = \infty$ $\boxed{\text{diverges}}$

$\boxed{\text{Bounded below by } \frac{1}{2}}$

Monotonic: $\frac{1}{2} < \frac{4}{3} < \frac{9}{4} < \frac{16}{5}$

$\frac{n^2}{n+1}$ $\quad \frac{(n+1)^2}{(n+1)+1} = \frac{n^2+2n+1}{n+2}$

$\frac{n^2}{n+1} < \frac{n^2+2n+1}{n+2}$

increasing

$\boxed{\text{monotonic}}$

Absolute Value Theorem:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

This only works if the answer $= 0$

Example 7:

Show that $\lim_{n\to\infty} a_n = 0$ if $\{a_n\} = \left\{\frac{(-1)^n n}{3^n}\right\}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n n}{3^n}\right| = \lim_{n\to\infty} \frac{n}{3^n} = \lim_{n\to\infty} \frac{1}{(\ln 3)(3^n)} = \frac{1}{(\ln 3)(3^\infty)} = \frac{1}{\infty} = 0$$

↑ L'Hopitals Rule

$\boxed{\lim_{n\to\infty} a_n = 0}$ ✓