AP Calculus BC
Unit 8 – Review

Name: Answer Key *

Evaluate each integral. Show your work on separate paper.

1) $\int \frac{3x}{\sqrt{4-9x^2}}\,dx$ $\quad -\frac{1}{3}\sqrt{4-9x^2}+C$

2) $\int x^3 e^{2x}\,dx$
$\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}xe^{2x}-\frac{3}{8}e^{2x}+C$

3) $\int \sec^3 x\tan^3 x\,dx$
$\frac{1}{5}\sec^5 x-\frac{1}{3}\sec^3 x+C$

4) $\int \ln 5x\,dx$
$x\ln(5x)-x+C$

5) $\int \sin^5 x\cos^4 x\,dx$
$-\frac{1}{5}\cos^5 x+\frac{2}{7}\cos^7 x-\frac{1}{9}\cos^9 x+C$

6) $\int \frac{\sec x}{\tan^2 x}\,dx$
$-\csc x+C$

7) $\int \sin^2 x\,dx$
$\frac{1}{2}x-\frac{1}{4}\sin 2x+C$

8) $\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$
$2\sqrt{\tan x}+C$

9) $\int \frac{5}{x^3-x}\,dx$
$-5\ln|x|+5/2\ln|x+1|+5/2\ln|x-1|+C$

10) $\int e^{2x}\sin x\,dx$
$\frac{2}{5}e^{2x}\sin x-\frac{1}{5}e^{2x}\cos x+C$

11) Solve the differential equation:
$\frac{dy}{dt}=0.3y(4-y),\ \ y(0)=1$
$y=\frac{4}{1+3e^{-1.2t}}$

Evaluate the improper integrals. Show your work!

12) $\int_0^\infty e^{3x}\,dx$ $\quad$ diverges

13) $\int_{-1}^{2}\frac{dx}{x^3}$ $\quad$ diverges

(Calculator Active)
14) Twenty-eight lowland gorillas were known to be in a wild animal preserve in 1970. The rate of growth of this population is $\frac{dP}{dt}=P(0.1-0.0004P)$, where time t is in years.

a) What is $\lim_{t\to\infty}P(t)$? Interpret this limit in the context of the problem. 250 gorillas

b) What is the rate of change of the population of gorillas when it is growing fastest? Indicate units of measure.
125 gorillas per year

c) Solve the differential equation, given that $P(0)=28$. Write your answer so that P is a function of t.
$P=\frac{250}{1+7.929e^{-0.1t}}$

(1) $$\frac{-1}{18}\cdot 3\int \frac{\overset{-18}{\cancel{3}}x}{\sqrt{4-9x^2}}\,dx$$

$$u = 4-9x^2$$
$$du = -18x\,dx$$

$$-\frac{1}{6}\int \frac{1}{\sqrt{u}}\,du$$

$$-\frac{1}{6}\int u^{-1/2}\,du$$

$$-\frac{1}{6}\left[\frac{2}{1}u^{1/2}\right] + c$$

$$-\frac{1}{3}(4-9x^2)^{1/2} + c \qquad \boxed{-\frac{1}{3}\sqrt{4-9x^2} + c}$$

(2) $$\int x^3e^{2x}\,dx$$

$$u = x^3 \qquad v = \frac{1}{2}e^{2x}$$
$$du = 3x^2\,dx \qquad dv = e^{2x}\,dx$$

$$(x^3)(\tfrac{1}{2}e^{2x}) - \int (\tfrac{1}{2}e^{2x})(3x^2)\,dx$$

$$-\frac{3}{2}\int x^2e^{2x}\,dx$$

$$u = x^2 \qquad v = \frac{1}{2}e^{2x}$$
$$du = 2x\,dx \qquad dv = e^{2x}\,dx$$

$$-\frac{3}{2}\left[(x^2)(\tfrac{1}{2}e^{2x}) - \int (\tfrac{1}{\cancel{2}}e^{2x})(\cancel{2}x)\,dx\right]$$

$$-\frac{3}{4}x^2e^{2x} + \frac{3}{2}\int e^{2x}x\,dx$$

$$u = x \qquad v = \frac{1}{2}e^{2x}$$
$$du = dx \qquad dv = e^{2x}\,dx$$

$$\frac{3}{2}\left[(x)(\tfrac{1}{2}e^{2x}) - \int (\tfrac{1}{2}e^{2x})\,dx\right)$$

$$\frac{3}{4}xe^{2x} - \frac{3}{2}\int \frac{1}{2}e^{2x}\,dx$$

$$-\frac{3}{4}\int e^{2x}\,dx$$

$$-\frac{3}{4}\left(\frac{1}{2}e^{2x}\right) + c$$

$$\boxed{\frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + c}$$

③ $\int \sec^3 x \tan^3 x \, dx$

$\dfrac{\sin^2 x + \cos^2 x = 1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\int \sec^2 x \tan^2 x \sec x \tan x \, dx$

$\int \sec^2 x (\sec^2 x - 1)(\sec x \tan x) dx$

$\int (\sec^4 x - \sec^2 x)(\sec x \tan x) dx$

$\int u^4 - u^2 \, du$

$u = \sec x$
$du = \sec x \tan x \, dx$

$\frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$\boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

④ $\int \ln 5x \, dx$

$u = \ln 5x \qquad v = x$
$du = \dfrac{5}{5x} dx \qquad dv = 1 dx$

$(\ln 5x)(x) - \int (x)\left(\frac{5}{5x}\right) dx$

$-\int dx$

$-x$

$\boxed{x \ln 5x - x + C}$

$\sin^2 x + \cos^2 x = 1$

(5) $\int \sin^{5} x \cos^4 x \, dx$ odd

$\int \sin^4 x \cos^4 x \sin x \, dx$

$\int \underbrace{(1-\cos^2 x)^2}\,(\cos^4 x)\,(\sin x)\,dx$

$\int \underbrace{(1-2\cos^2 x+\cos^4 x)(\cos^4 x)}\,(\sin x)\,dx$

$-1\int^{-1}(\cos^4 x - 2\cos^6 x + \cos^8 x)\sin x\,dx$

$-1\int u^4 - 2u^6 + u^8\,du$ $\quad u = \cos x$, $du = -\sin x\,dx$

$-1\left[\frac{1}{5}u^5 - 2\left(\frac{1}{7}u^7\right) + \frac{1}{9}u^9\right] + C$

$\boxed{-\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C}$

(6) $\int \frac{\sec x}{\tan^2 x}\,dx$

$\int \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}}$ $\quad \frac{1}{\cancel{\cos x}} \cdot \frac{\cos^{\cancel{2}} x}{\sin^2 x} = \frac{\cos x}{\sin^2 x}$

$\int \frac{\cos x}{\sin^2 x}\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$\int \frac{du}{u^2} = \int u^{-2}\,du = -1u^{-1} + C$

$-1\sin^{-1} + C$

$\frac{-1}{\sin} + C = \boxed{-\csc x + C}$

⑦ $\int \sin^2 x\,dx$

$\int \frac{1}{2}(1-\cos 2x)\,dx$

$\int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$\int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 2x\,dx$

$\frac{1}{2}x \quad - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]$

$-\frac{1}{4}\sin 2x$

$\boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$

⑧ $\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx \qquad u = \tan x$

$du = \sec^2 x\,dx$

$\int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$

$= \frac{2}{1}u^{1/2} + C$

$= 2(\tan x)^{1/2} + C$

$= \boxed{2\sqrt{\tan x} + C}$

(9) $\int \frac{5}{x^3-x}\,dx$

$$x^3-x = x(x^2-1) = x(x+1)(x-1)$$

$$\frac{5}{x^3-x} = \frac{A^{(x+1)(x-1)}}{x_{(x+1)(x-1)}} + \frac{B^{x(x-1)}}{x+1_{x(x-1)}} + \frac{C^{x(x+1)}}{x-1_{x(x+1)}}$$

$$5 = A(x+1)(x-1) + B(x)(x-1) + C(x)(x+1)$$

$\underline{x=-1}$: $5 = A(\cancel{-1+1})(-1-1) + B(-1)(-1-1) + C\cancel{(-1)(-1+1)}$

$5 = B(-1)(-2)$

$5 = 2B \qquad B = 5/2$

$\underline{x=0}$: $5 = A(0+1)(0-1) + B(0)\cancel{(0-1)} + C(0)\cancel{(0+1)}$

$5 = A(1)(-1)$

$5 = -A \qquad A = -5$

$\underline{x=1}$: $5 = A\cancel{(1+1)}(1-1) + B\cancel{(1)(1-1)} + C(1)(1+1)$

$5 = 2C \qquad C = 5/2$

$$\int \frac{-5}{x}\,dx + \int \frac{5/2}{x+1}\,dx + \int \frac{5/2}{x-1}\,dx$$

$$\boxed{-5\ln|x| + 5/2\ln|x+1| + 5/2\ln|x-1| + C}$$

(10) $\int e^{2x}\sin x\,dx$

$u = \sin x \qquad v = \frac{1}{2}e^{2x}$

$du = \cos x\,dx \qquad dv = e^{2x}\,dx$

$(\sin x)(\frac{1}{2}e^{2x}) - \int (\frac{1}{2}e^{2x})(\cos x)dx$

$u = \cos x \qquad v = \frac{1}{4}e^{2x}$

$du = -\sin x\,dx \qquad dv = \frac{1}{2}e^{2x}dx$

$-\left[(\cos x)(\frac{1}{4}e^{2x}) - \int (\frac{1}{4}e^{2x})(-\sin x)dx\right]$

$-\frac{1}{4}\cos x e^{2x} + \int (\frac{1}{4}e^{2x})(-\sin x)dx$

$-\frac{1}{4}\int e^{2x}\sin x\,dx$

$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$

$\frac{\frac{5}{4}\int e^{2x}\sin x\,dx}{5/4} = \frac{\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x}{5/4}$

$\frac{1}{2}\cdot\frac{4}{5} = \frac{4}{10}$

$\int e^{2x}\sin x\,dx = \boxed{\frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C}$

$-\frac{1}{4}\cdot\frac{4}{5}$

(11) $\frac{dy}{dt} = 0.3y(4-y) \quad y(0)=1$

$$\int \frac{dy}{y(4-y)} = \int 0.3\,dt$$

$$\frac{1}{y(4-y)} = \frac{A}{y}\,(4-y) + \frac{B}{4-y}\,(y)$$

$$1 = A(4-y) + B(y)$$

$\underline{y=4}$: $1 = \cancel{A(4-4)} + B(4)$

$B = 1/4$

$\underline{y=0}$: $1 = A(4-0) + \cancel{B(0)}$

$A = 1/4$

$$\int \frac{1/4}{y} + \frac{1/4}{4-y}\,dy = \int 0.3\,dt$$

$$\tfrac{1}{4}\ln|y| - \tfrac{1}{4}\ln|4-y| = 0.3t + C$$

$$\frac{\frac{1}{4}(\ln y - \ln(4-y))}{1/4} = \frac{0.3t + C}{1/4}$$

$$\frac{3}{10} \cdot \frac{4}{1} = \frac{12}{10}$$

$$\ln y - \ln(4-y) = 1.2t + C$$

$$-\ln y + \ln(4-y) = -1.2t - C$$

$$\cancel{e^{\ln}}\left(\frac{4-y}{y}\right) = e^{-1.2t-C}$$

$$\frac{4-y}{y} = e^{-1.2t}e^{-C}$$

$$\frac{4}{y} - 1 = e^{-1.2t}e^{-C}$$

$$\frac{4}{y} = 1 + e^{-1.2t}e^{-C}$$

$y(0)=1$

$$\frac{4}{1} = 1 + \cancel{e^{-1.2(0)}}e^{-C}$$

$$4 = 1 + e^{-C}$$

$$e^{-C} = 3$$

$$\frac{4}{y} = 1 + 3e^{-1.2t}$$

$$\boxed{y = \frac{4}{1+3e^{-1.2t}}}$$

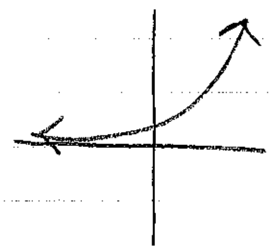
(12) $\int_0^{\infty} e^{3x}\,dx$

$\lim_{b\to\infty}\int_0^b e^{3x}\,dx$

$\left[\frac{1}{3}e^{3x}\right]_0^b$

$\frac{1}{3}e^{3(b)} - \frac{1}{3}e^{3(0)}$

$\lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}\right]$



$\infty - \frac{1}{3}$

$\boxed{\text{diverges}}$

(13) $\int_{-1}^{2} \frac{dx}{x^3}$ $\quad x \neq 0$

$\int_{-1}^{0} \frac{1}{x^3}\,dx + \int_0^2 \frac{1}{x^3}\,dx$

$\lim_{b\to 0}\int_{-1}^{b} x^{-3}\,dx$

$\left[-\frac{1}{2}x^{-2}\right]_{-1}^{b}$

$-\frac{1}{2}(b)^{-2} - \frac{1}{2}(-1)^{-2}$

$\lim_{b\to 0}\left[\frac{-1}{2b^2} - \frac{1}{2}\right]$

$\frac{-1}{0} - \frac{1}{2}$ $\quad \boxed{\text{diverges}}$

DNE

(14) $\frac{dP}{dt} = P(\frac{0.1}{.0004} - \frac{.0004P}{.0004})$

$\frac{dP}{dt} = .0004P(250 - P)$

a) $\lim_{t \to \infty} P(t) = \boxed{250}$ $\boxed{\text{The wild animal park can support 250 gorillas.}}$

b) $\frac{250}{2} = \boxed{125 \text{ gorillas per year}}$

c) $\frac{dP}{dt} = .0004P(250-P)$

$\int \frac{dP}{P(250-P)} = \int .0004\,dt$

$\frac{1}{P(250-P)} = \frac{A}{P} + \frac{B}{250-P}$

$1 = A(250-P) + B(P)$

$P=250$: $1 = A(\cancel{250-250}) + B(250)$

$B = 1/250$

$P=0$: $1 = A(250-0) + \cancel{B(0)}$

$A = 1/250$

$\int \frac{1/250}{P} + \frac{1/250}{250-P}\,dP = \int .0004dt$

$\frac{1}{250}\ln|P| - \frac{1}{250}\ln|250-P| = .0004t + C$

$\frac{1}{250}(\ln P - \ln(250-P)) = .0004t + C$

$\ln P - \ln(250-P) = 0.1t + C$

$-\ln P + \ln(250-P) = -.1t - C$

$\cancel{e^{\ln}}\left(\frac{250-P}{P}\right) = e^{-0.1t - C}$

$\frac{250}{P} - 1 = e^{-0.1t}e^{-C}$

$\frac{250}{P} = 1 + e^{-0.1t}e^{-C}$

$P(0) = 28$

$\frac{250}{28} = 1 + e^{-0.1(0)}e^{-C}$

$8.929 = 1 + e^{-C}$

$7.929 = e^{-C}$ $\rightarrow$ $\frac{250}{P} = 1 + 7.929e^{-0.1t}$

$\boxed{P = \frac{250}{1 + 7.929e^{-0.1t}}}$