AP Calculus BC
Unit 8 – Integration Techniques

Day 9 Notes: Improper Integrals

Improper integrals occur when:
-one (or both) limit of integration is ∞
-there is a discontinuity in the integrand somewhere between the limits of integration

CONTINUOUS FUNCTIONS

1. **f (x) is continuous on [a, ∞)⇨**

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Example 1:

$$\int_0^{\infty} xe^{-x}dx$$

$u = x \quad v = -e^{-x}$
$du = dx \quad dv = e^{-x}dx$

$(x)(-e^{-x}) - \int(-e^{-x})(dx)$
$+\int e^{-x}dx$
$-e^{-x}$

$$\lim_{b\to\infty}\int_0^b xe^{-x}dx$$

$$\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b$$

$$\lim_{b\to\infty}\left[-be^{-b}-e^{-b}+\cancel{0e^{-0}}+e^{-0}\right] = \lim_{b\to\infty}\left[\frac{-b}{e^b}-\frac{1}{e^b}+1\right] = 0-0+1$$

$$=\boxed{1}$$

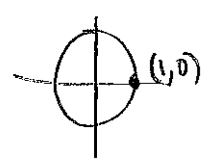
$\boxed{\text{converges}}$

2. **If f(x) is continuous on (-∞, b], then**

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

Example 2:

$$\int_{-\infty}^{0} sin\frac{x}{2}dx$$



$$\lim_{b\to-\infty}\int_b^0 \sin\left(\frac{x}{2}\right)dx$$

$$\lim_{b\to-\infty}\left[-2\cos\left(\frac{x}{2}\right)\right]_b^0 = \lim_{b\to-\infty}\left[-2\underbrace{\cos\left(\frac{0}{2}\right)}_{(1)}+2\cos\left(\frac{b}{2}\right)\right]$$

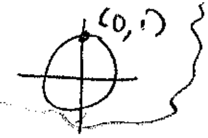
$\lim_{x\to\infty}\cos(x) = \text{DNE}$
b/c oscillates

$$\lim_{b\to-\infty}\left[-2+2\cos\left(\frac{b}{2}\right)\right] = -2+\text{DNE}$$

$$=\boxed{\text{DNE}}$$

$\boxed{\text{diverges}}$

$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$ b/c As $x \to \frac{\pi}{2}$, $\tan x \to \infty$



$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

$$= \lim_{a\to-\infty}\int_{a}^{c} f(x)dx + \lim_{b\to\infty}\int_{c}^{b} f(x)dx$$

Example 3:

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx$$

$\int \frac{e^x}{1+e^{2x}}dx$

$\int \frac{du}{a^2+u^2}$ $\quad u = e^x$, $du = e^x dx$, $a = 1$

$\frac{1}{1}\arctan e^x + C$

$$\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}dx$$

$\lim_{b\to-\infty} \int_b^0 \frac{e^x}{1+e^{2x}}dx$

$\lim_{b\to-\infty} \left[\arctan e^x\right]_b^0$

$\lim_{b\to-\infty} \left[\arctan e^0 - \arctan e^b\right]$

$\lim_{b\to-\infty} \left[\arctan 1 - \arctan e^b\right]$

$\lim_{b\to-\infty} \left[\frac{\pi}{4} - \arctan e^b\right]$

$\frac{\pi}{4} - \arctan e^{-\infty}$

$\frac{\pi}{4} - \arctan 0$

$\frac{\pi}{4} - 0$

$\lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}}dx$

$\lim_{b\to\infty} \left[\arctan e^x\right]_0^b$

$\lim_{b\to\infty} \left[\arctan e^b - \arctan \cancel{e^0}^{\,1}\right]$

$\lim_{b\to\infty} \left[\arctan e^b - \frac{\pi}{4}\right]$

$\arctan e^{\infty} - \frac{\pi}{4}$

$\arctan \infty - \frac{\pi}{4}$

$\frac{\pi}{2} - \frac{\pi}{4}$

$\frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$ converges

DISCONTINUOUS FUNCTION

1. If f is continuous on [a, b), then

$$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

Example 4:

$x \neq \pi/2$

$$\int_0^{\pi/2} tanx dx$$

$$\lim_{c\to\frac{\pi}{2}^-}\int_0^b \tan x\, dx$$

$$\lim_{c\to\frac{\pi}{2}^-}\left[-\ln|\cos x|\right]_0^b$$

$$\lim_{c\to\frac{\pi}{2}^-}\left[-\ln|\cos b| - \ln|\cos 0|\right]$$

$$\lim_{c\to\frac{\pi}{2}^-}\left[-\ln|\cos b| - \ln 1\right] = \lim_{c\to\frac{\pi}{2}^-}\left[-\ln|\cos b|\right]$$

$$= -\ln\left|\cos\frac{\pi}{2}\right|$$

$$= -\ln|0|$$

$$= \boxed{DNE} \quad \boxed{\infty} \quad \boxed{\text{diverges}}$$

2. If f is continuous on (a, b], then

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

Example 5:

$x \neq 0$

$$\int_0^2 \frac{1}{x^2}dx$$

$$\lim_{c\to 0^+}\int_c^2 x^{-2}dx$$

$$\lim_{c\to 0^+}\left[-x^{-1}\right]_c^2$$

$$\lim_{c\to 0^+}\left[-\frac{1}{2}+\frac{1}{c}\right] = -\frac{1}{2}+\cancel{\frac{1}{0}} \quad \boxed{DNE} \quad \boxed{\infty} \quad \boxed{\text{diverges}}$$

3. If f is continuous on [a, b] except at x = c, where there is an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{c\to d^-}\int_a^d f(x)\,dx + \lim_{c\to d^+}\int_d^b f(x)\,dx$$

Example 6:

$x \neq 1$

$$\int_0^2 \frac{1}{(x-1)^2}\,dx$$

$$\int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$\lim_{c\to 1^-}\int_0^c (x-1)^{-2}\,dx + \lim_{c\to 1^+}\int_c^0 (x-1)^{-2}\,dx$$

$$\left[-\frac{1}{x-1}\right]_0^c$$

This side will also diverge.

$$\lim_{c\to 1}\left[\frac{-1}{x-1} + \frac{1}{0-1}\right]$$

$$\frac{-1}{\cancel{1-1}} + \frac{1}{0-1} = \infty - 1 = \boxed{\text{diverges}}$$

A special type of improper integral:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \le 1 \end{cases}$$

Example 7:

special type

$$\int_1^\infty \frac{1}{x^3}\,dx$$

$p = 3$

$$\frac{1}{3-1} = \boxed{\frac{1}{2}}$$

Example 8:

special type

$$\int_1^\infty \frac{4}{\sqrt{x}}\,dx = 4\int_1^\infty \frac{1}{x^{1/2}}$$

$p = 1/2$

$\boxed{\text{diverges}}$