AP Calculus BC
Unit 8 – Integration Techniques

Day 8 Notes: Integration Using Partial Fractions

**Sometimes it is necessary to "decompose" a rational function into simpler expressions so that we can integrate. We usually decompose a rational function when the denominator is easy to factor.*

CASE 1: Two Linear Factors

$$\int \frac{1}{x^2+3x-18}\,dx$$

① Factor denominator.

$$x^2+3x-18 = (x+6)(x-3)$$

$$\int \frac{1}{x^2+3x-18}\,dx = \int \frac{A\,(x-3)}{x+6\,(x-3)} + \int \frac{B\,(x+6)}{x-3\,(x+6)}$$

② Multiply both sides by common denominator.

$$1 = A(x-3) + B(x+6)$$

③ Choose values of x that cause one term to zero out.

$x=3$: $1 = A(\cancel{3-3}) + B(3+6)$
$1 = 9B$
$\boxed{B = 1/9}$

$x=-6$: $1 = A(-6-3) + B(\cancel{-6+6})$
$1 = A(-9)$
$\boxed{A = -1/9}$

$$\int \frac{1}{x^2+3x-18}\,dx = \int \frac{-1/9}{x+6} + \int \frac{1/9}{x-3}$$

$$= -\frac{1}{9}\int \frac{1}{x+6}\,dx + \frac{1}{9}\int \frac{1}{x-3}\,dx$$

$u = x+6$
$du = 1\,dx$

$$-\frac{1}{9}\int \frac{du}{u} \qquad \frac{1}{9}\int \frac{du}{u}$$

$u = x-3$
$du = dx$

$$\boxed{-\frac{1}{9}\ln|x+6| + \frac{1}{9}\ln|x-3| + C}$$

$$\boxed{\frac{1}{9}\ln\left|\frac{x-3}{x+6}\right| + C}$$

CASE 2: Repeated Linear Factors

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx$$

① Factor denominator.

$x(x^2+2x+1) = x(x+1)(x+1) = x(x+1)^2$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x} = \int \frac{A}{x}\,\frac{(x+1)^2}{(x+1)^2} + \frac{B}{x+1}\,\frac{x(x+1)}{x(x+1)} + \frac{C}{(x+1)^2}\,\frac{(x)}{(x)}$$

$5x^2+20x+6 = A(x+1)^2 + B(x)(x+1) + C(x)$

$\underline{x=-1}$: $5(-1)^2+20(-1)+6 = A(\cancel{-1+1})^2 + B(-1)(\cancel{-1+1}) + C(-1)$

$5 - 20 + 6 = -C$

$-9 = -C$

$\boxed{C=9}$

$\underline{x=0}$: $5(\cancel{0})^2 + 2\cancel{0}(0)+6 = A(0+1)^2 + B(\cancel{0})(0+1) + C(\cancel{0})$

$\boxed{6 = A}$

any #

$\underline{x=1}$: $5(1)^2+20(1)+6 = A(1+1)^2 + B(1)(1+1) + C(1)$

$5+20+6 = A(4) + B(1)(2) + C$

$31 = 4A+2B+C$

plug in $A=6$ & $C=9 \rightarrow$ $31 = 4(6) + 2B + 9$

$-2 = 2B$

$\boxed{B=-1}$

$$\int \frac{6}{x}\,dx + \int \frac{-1}{x+1}\,dx + \int \frac{9}{(x+1)^2}\,dx$$

$u = x+1$
$du = dx$

$6\ln|x| + -\ln|x+1| +$ $9\int u^{-2}\,du$

$9[-u^{-1}]$

$\dfrac{-9}{(x+1)}$

$$\boxed{6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C}$$

CASE 3: Linear & Quadratic Factors

$$\int \frac{2x^3 - 4x - 8}{(x^3 - x)(x^2+4)}\,dx$$

$(x^3 - x)(x^2+4) = x(x-1)(x^2+4)$

$$\int \frac{2x^3-4x-8}{(x^3-x)(x^2+4)} = \int \frac{A}{x}\cdot\frac{(x-1)(x^2+4)}{(x-1)(x^2+4)} + \int \frac{B}{x-1}\cdot\frac{(x)(x^2+4)}{(x)(x^2+4)} + \int \frac{Cx+D}{x^2+4}\cdot\frac{(x)(x-1)}{(x)(x-1)}$$

$$2x^3-4x-8 = A(x-1)(x^2+4) + B(x)(x^2+4) + (Cx+D)(x)(x-1)$$

$\underline{x=1}$: $2(1)^3-4(1)-8 = \cancel{A(1-1)}(1^2+4) + B(1)(1^2+4) + \cancel{(C(1)+D)(1)(1-1)}$

$2-4-8 = B(1)(5)$

$-10 = 5B$

$\boxed{B=-2}$

$\underline{x=0}$: $2(0)^3-4(0)-8 = A(0-1)(0^2+4) + \cancel{B(0)(0^2+4)} + \cancel{(C(0)+D)(0)(0-1)}$

$-8 = A(-1)(4)$

$\boxed{A=2}$

any #

$\underline{x=2}$: $2(2)^3-4(2)-8 = A(2-1)(2^2+4) + B(2)(2^2+4) + (C(2)+D)(2)(2-1)$

$16-8-8 = A(1)(8) + B(2)(8) + (2C+D)(2)(1)$

$0 = 8A + 16B + (2C+D)(2)$

plug in $A=2$ & $B=-2$ $\rightarrow$ $0 = 8(2) + 16(-2) + (2C+D)(2)$

$0 = 16 - 32 + 4C + 2D$

$\boxed{16 = 4C + 2D}$

any #

$\underline{x=-1}$: $2(-1)^3-4(-1)-8 = A(-1-1)((-1)^2+4) + B(-1)((-1)^2+4) + (C(-1)+D)(-1)(-1-1)$

$-2+4-8 = A(-2)(5) + B(-1)(5) + (-C+D)(2)$

$-6 = -10A - 5B - 2C + 2D$

plug in $A=2$ & $B=-2$ $\rightarrow$ $-6 = -10(2) - 5(-2) - 2C + 2D$

$-6 = -20 + 10 - 2C + 2D$

$\boxed{4 = -2C + 2D}$

$16 = 4C + \cancel{2D}$

$\ominus\ 4 = -2C + \cancel{2D}$

$12 = 6C$

$\boxed{C=2}$

$16 = 4C + 2D$

$16 = 4(2) + 2D$

$8 = 2D$

$\boxed{D=4}$

$$\int \frac{2}{x}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{2x+4}{x^2+4}\,dx$$

$2\ln|x|$

$-2\ln|x-1|$

$\int \frac{2x}{x^2+4} + \int \frac{4}{x^2+4}$ $\quad u = 2x$

$\int \frac{du}{u}$

$\ln|x^2+4|$

$a=2$, $u=x$, $du=dx$

$4\int \frac{du}{a^2+u^2}$

$4\left(\frac{1}{2}\right)\arctan\frac{x}{2}$

$2\arctan\frac{x}{2}$

$$\boxed{2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\arctan\frac{x}{2} + C}$$

CASE 4: Repeated Quadratic Factors

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \int \frac{Ax+B}{x^2+2}\,(x^2+2) + \int \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx+D$$

$$8x^3+13x = \underbrace{Ax^3+2Ax+Bx^2+2B+Cx+D}_{\text{Put in descending order}}$$

$$\boxed{8x^3} + 13x = \boxed{Ax^3} + Bx^2 + x(2A+C) + (2B+D)$$

Compare each term

$\boxed{A=8}$ $\boxed{B=0}$

$13=2A+C$
$13=2(8)+C$
$13=16+C$
$\boxed{C=-3}$

$0=2B+D$
$0=2(0)+D$
$\boxed{D=0}$

$$\tfrac{1}{2}\cdot 8\int \frac{2\,\cancel{8}x}{x^2+2}\,dx + -3\int \frac{2\,\cancel{-3}x}{(x^2+2)^2}\,dx \quad \tfrac{1}{2}$$

$u=x^2+2$
$du=2x\,dx$

$u=x^2+2$
$du=2x\,dx$

$$4\int \frac{du}{u}$$

$$4\ln|x^2+2|$$

$$-\frac{3}{2}\int \frac{du}{u^2}$$

$$-\frac{3}{2}\int u^{-2}\,du$$

$$-\frac{3}{2}\left[-1u^{-1}\right]$$

$$\frac{+3}{2(x^2+2)}$$

$$\boxed{4\ln|x^2+2| + \frac{3}{2(x^2+2)}\,dx}$$