AP Calculus BC
Unit 8 – Day 8 – Assignment

Name: Answer Key

Use partial fractions to evaluate the integral.

1) $\int \frac{3}{x^2 + x - 2}\,dx$

$-\ln|x+2| + \ln|x-1| + C$

or

$\ln\left|\frac{x-1}{x+2}\right| + C$

2) $\int \frac{5 - x}{2x^2 + x - 1}\,dx$

$\frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$

3) $\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

$3\ln|x| + \frac{1}{x} + \ln|x+1| + C$

4) $\int \frac{x^2 - 1}{x^3 + x}\,dx$

$-\ln|x| + \ln|x^2+1| + C$

or

$\ln\left|\frac{x^2+1}{x}\right| + C$

5) $\int \frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)}\,dx$

$\ln|x+1| + \frac{2}{\sqrt{2}}\arctan\frac{x-1}{\sqrt{2}} + C$

① $\int \frac{3}{x^2+x-2}\,dx$ $\qquad x^2+x-2=(x+2)(x-1)$

$$\frac{3}{x^2+x-2} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

$$\frac{A}{(x+2)}\cdot\frac{(x-1)}{(x-1)} \qquad \frac{B}{(x-1)}\cdot\frac{(x+2)}{(x+2)}$$

$3 = A(x-1) + B(x+2)$

$x=1$: $3 = A(1-1) + B(1+2)$

$3 = 3B$

$\boxed{B=1}$

$x=-2$: $3 = A(-2-1) + B(-2+2)$

$3 = A(-3)$

$\boxed{A=-1}$

$$\int \frac{-1}{x+2}\,dx + \int \frac{1}{x-1}\,dx$$

$$\boxed{-\ln|x+2| + \ln|x-1| + C}$$

$$\boxed{\ln\left|\frac{x-1}{x+2}\right| + C}$$

② $\int \frac{5-x}{2x^2+x-1}\,dx$

$2x^2+x-1 = (2x-1)(x+1)$

$2 \quad -2 \quad -1$

$(2x^2+2x)(-1x-1)$

$2x(x+1) \quad -1(x+1)$

$$\frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1}\ {}^{(x+1)}_{(x+1)} + \frac{B}{x+1}\ {}^{(2x-1)}_{(2x-1)}$$

$5-x = A(x+1) + B(2x-1)$

$x=-1$: $5-(-1) = A(-1+1) + B(2(-1)-1)$

$6 = B(-3)$

$\boxed{B=-2}$

$x=\frac{1}{2}$: $5-\frac{1}{2} = A(\frac{1}{2}+1) + B(2(\frac{1}{2})-1)$

$\frac{9}{2} = A(\frac{3}{2})$

$\frac{9}{2}\cdot\frac{2}{3}$ $\quad \boxed{A=3}$

$$\tfrac{1}{2}\cdot 3\int \frac{2}{2x-1}\,dx + \int \frac{-2}{x+1}\,dx$$

$u = 2x-1$

$du = 2dx$

$\frac{3}{2}\int \frac{du}{u}$

$$\boxed{\tfrac{3}{2}\ln|2x-1| + -2\ln|x+1| + C}$$

③ $\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$

$x^3+x^2 = x^2(x+1)$

$x(x)(x+1)$ — repeated

$$\frac{4x^2+2x-1}{x^3+x^2} = \frac{A}{x}\ {}^{(x)(x+1)} + \frac{B}{x^2}\ {}^{(x+1)} + \frac{C}{x+1}\ {}^{(x^2)}$$] Case #2

$4x^2+2x-1 = A(x)(x+1) + B(x+1) + C(x^2)$

$x=0$: $4(0)^2+2(0)-1 = A(0)(0+1)+B(0+1)+C(0^2)$

$-1 = B$

$x=-1$: $4(-1)^2+2(-1)-1 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$

$4-2-1 = C$

$1 = C$

any # ↓

$x=1$: $4(1)^2+2(1)-1 = A(1)(1+1)+B(1+1)+C(1^2)$

$4+2-1 = A(2)+B(2)+C$

$5 = 2A+2B+C$

plug in $B=-1$ & $C=1$ → $5 = 2A+2(-1)+1$

$6 = 2A$

$A=3$

$$\int \frac{3}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{1}{x+1}\,dx$$

$3\ln|x|$; $-\int x^{-2}\,dx$; $-[-x^{-1}]$; $\frac{1}{x}$; $\ln|x+1|$

$$3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$

④ $\int \frac{x^2-1}{x^3+x}\,dx$ $\qquad x^3+x = x(x^2+1)$

$$\frac{x^2-1}{x^3+x} = \frac{A}{x}\ {}^{(x^2+1)} + \frac{Bx+C}{x^2+1}\ {}^{(x)}$$

$$x^2-1 = A(x^2+1) + (Bx+C)(x)$$

$x=0$: $0^2-1 = A(0^2+1) + (B(0)+C)(0)$

$$-1 = A$$

Any # $x=1$: $1^2-1 = A(1^2+1) + (B(1)+C)(1)$

$0 = 2A + B + C$

plug in $A=-1$ → $0 = 2(-1) + B + C$

$2 = B + C$

Any # $x=2$: $2^2-1 = A(2^2+1) + (B(2)+C)(2)$

$3 = 5A + 4B + 2C$

plug in $A=-1$ → $3 = 5(-1) + 4B + 2C$

$8 = 4B + 2C$

$2 = B + C$ (mult by 2) → $4 = 2B + 2C$

$8 = 4B + 2C$

$-4 = -2B$

$B = 2$

$2 = B + C$

$2 = 2 + C$

$C = 0$

$$\int \frac{-1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$-\ln|x| + \ln|x^2+1| + C$$

$$\ln\left|\frac{x^2+1}{x}\right| + C$$

⑤ $\int \frac{x^2+5}{(x+1)(x^2-2x+3)}\,dx$

$$\frac{x^2+5}{(x+1)(x^2-2x+3)} = \frac{A}{x+1}\,\frac{(x^2-2x+3)}{(x^2-2x+3)} + \frac{Bx+C}{x^2-2x+3}\,\frac{(x+1)}{(x+1)}$$

$$x^2+5 = A(x^2-2x+3) + (Bx+C)(x+1)$$

$x=-1$: $(-1)^2+5 = A((-1)^2-2(-1)+3) + (B(-1)+C)(-1+1)$

$6 = A(1+2+3)$

$6 = 6A$

$\boxed{A=1}$

★ Compare each side of eqn:

$$x^2+5 = Ax^2 - 2Ax + 3A + Bx^2 + Bx + Cx + C$$

$$\boxed{x^2} + 0x + \textcircled{5} = \boxed{x^2(A+B)} + x(-2A+B+C) + \textcircled{(3A+C)}$$

$1 = A+B$

plug in $A=1$ → $1 = 1+B$

$\boxed{B=0}$

$5 = 3A+C$

plug in $A=1$ ↘ $5 = 3(1)+C$

$\boxed{2=C}$

$$\int \frac{1}{x+1}\,dx + \int \frac{2}{x^2-2x+3}\,dx$$

Complete the square:

$x^2-2x+\left(\frac{-2}{2}\right)^2+3-\left(\frac{-2}{2}\right)^2$

$(x-1)^2+2$

$\textcircled{\ln|x+1|}$

$$2\int \frac{1}{(x-1)^2+2}$$

$u = x-1$

$du = dx$

$a = \sqrt{2}$

$$2\int \frac{du}{u^2+a^2}$$

$$2\left(\frac{1}{\sqrt{2}}\right)\arctan\left(\frac{x-1}{\sqrt{2}}\right)$$

$$\boxed{\ln|x+1| + \frac{2}{\sqrt{2}}\arctan\frac{x-1}{\sqrt{2}} + C}$$