AP Calculus BC
Unit 8 – Integration Techniques

Day 7 Notes: Trig Integrals with Powers of Secant & Tangent

$$\int sec^m(ax)tan^n(ax)dx$$

CASE 1: EVEN POWER OF SECANT

Save sec^2x and convert the rest to tanx using the identity $tan^2x + 1 = sec^2x$

Example 1:

$$\int sec^{\overset{even}{4}}x\sqrt{tanx}dx$$

$$\int (sec^2x)(tanx)^{1/2}(sec^2x)dx$$

$$\int (tan^2x+1)(tanx)^{1/2}(sec^2x)dx$$

$$\int [(tanx)^{5/2}+(tanx)^{1/2}](sec^2x)dx$$

$u=tanx$
$du=sec^2x\,dx$

$$\int u^{5/2}+u^{1/2}du$$

$$\frac{2}{7}u^{7/2}+\frac{2}{3}u^{3/2}+C$$

$$\boxed{\frac{2}{7}(tanx)^{7/2}+\frac{2}{3}(tanx)^{3/2}+C}$$

CASE 2: ODD POWER OF TANGENT

Save secxtanx and convert the rest to secx using the identity $tan^2x + 1 = sec^2x$

Example 2:

$$\int tan^{\overset{odd}{3}}\left(\frac{\pi x}{2}\right)\cdot sec^3\left(\frac{\pi x}{2}\right)dx$$

$$\int tan^2\left(\tfrac{\pi x}{2}\right)\cdot sec^2\left(\tfrac{\pi x}{2}\right)\cdot tan\left(\tfrac{\pi x}{2}\right)\cdot sec\left(\tfrac{\pi x}{2}\right)dx$$

$$\int \underbrace{\left(sec^2\left(\tfrac{\pi x}{2}\right)-1\right)\cdot sec^2\left(\tfrac{\pi x}{2}\right)}\cdot tan\left(\tfrac{\pi x}{2}\right)\cdot sec\left(\tfrac{\pi x}{2}\right)dx$$

$$\frac{2}{\pi}\int\left[sec^4\left(\tfrac{\pi x}{2}\right)-sec^2\left(\tfrac{\pi x}{2}\right)\right]\frac{\pi}{2}tan\left(\tfrac{\pi x}{2}\right)sec\left(\tfrac{\pi x}{2}\right)dx$$

$u=sec(\frac{\pi x}{2})$
$du=\frac{\pi}{2}sec(\frac{\pi x}{2})tan(\frac{\pi x}{2})dx$

$$\frac{2}{\pi}\int u^4-u^2\,du$$

$$\frac{2}{\pi}\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]+C$$

$$\frac{2}{5\pi}u^5-\frac{2}{3\pi}u^3+C$$

$$\boxed{\frac{2}{5\pi}sec^5\left(\tfrac{\pi x}{2}\right)-\frac{2}{3\pi}sec^3\left(\tfrac{\pi x}{2}\right)+C}$$

CASE 3: EVEN POWER OF TANGENT, NO SECANT

Convert one $\tan^2 x$ to $(\sec^2 x - 1)$. Repeat if necessary.

Example 3:

$$\int \tan^4 x dx$$

$$\int (\tan^2 x)(\tan^2 x) dx$$

$$\int (\sec^2 x - 1)(\tan^2 x) dx$$

$$\int \sec^2 x \tan^2 x - \int \tan^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x dS$

$$\int u^2 du \qquad - \int \sec^2 x - 1 \, dx$$

$$\tfrac{1}{3}u^3 \qquad - \int \sec^2 x dx + \int 1 dx$$

$$\tfrac{1}{3}\tan^3 x \qquad -\tan x + x$$

$$\boxed{\tfrac{1}{3}\tan^3 x - \tan x + x + C}$$

CASE 4: ODD POWER OF SECANT, NO TANGENT

Use integration by parts with $dv = \sec^2 x$.

Example 4:

$$\int \sec^3 x dx$$

$$\int \sec^2 x \sec x \, dx$$

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x \qquad dv = \sec^2 x \, dx$

$$(\sec x)(\tan x) - \int (\tan x)(\sec x \tan x) dx$$

$$- \int \tan^2 x \sec x \, dx$$

$$- \int (\sec^2 x - 1)(\sec x) \, dx$$

$$- \int \sec^3 x + \int \sec x \, dx$$

$$+ \ln|\sec x + \tan x|$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x + \ln|\sec x + \tan x|$$

$$2\int \sec^3 x dx = \sec x \tan x + \ln|\sec x + \tan x|$$

CASE 5: *When none of these work, convert everything to sines and cosines!!*

$$\boxed{\int \sec^3 x \, dx = \tfrac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C}$$