AP Calculus BC
Unit 8 – Integration Techniques

Day 6 Notes: Trig Integrals with Powers of Sine & Cosine

$$\int sin^m(ax)cos^n(ax)dx$$

CASE 1: ODD POWER OF SINE

Save one factor of sine, then convert the rest to cosine using the identity
$sin^2x = 1 - cos^2x$

Example 1:

$$\int sin^{3}x\,dx$$

odd

$\int (sin^2x)(sin x)\,dx$

$\int (1-cos^2x)(sin x)\,dx$

$\int sin x - cos^2x\, sin x\, dx$

$\int sin x + \int -cos^2x\, sin x\, dx$

$-cos x$

$u = cos x$
$du = -sin x\, dx$

$\int u^2\,du$

$\frac{1}{3}u^3 + C$

$\frac{1}{3}cos^3x + C$

$\boxed{-cos x + \frac{1}{3}cos^3x + C}$

Example 2:

$$\int sin^{5}(4x)\cdot cos(4x)dx$$

odd

$\int sin^4(4x)\, cos(4x)\cdot sin(4x)\,dx$

$\int (1-cos^2(4x))^2\cdot cos(4x)\cdot sin(4x)\,dx$

$\int (1-2cos^2(4x)+cos^4(4x))\cdot cos(4x)\cdot sin(4x)\,dx$

$-\frac{1}{4}\int [cos(4x) - 2cos^3(4x) + cos^5(4x)]\, sin(4x)\,dx$

$u = cos 4x$
$du = -4 sin 4x$

$-\frac{1}{4}\int u - 2u^3 + u^5\,du$

$-\frac{1}{4}\left[\frac{1}{2}u^2 - 2\left(\frac{1}{4}u^4\right) + \frac{1}{6}u^6\right] + C$

$-\frac{1}{8}u^2 + \frac{1}{8}u^4 - \frac{1}{24}u^6 + C$

$\boxed{-\frac{1}{8}cos^2(4x) + \frac{1}{8}cos^4(4x) - \frac{1}{24}cos^6(4x) + C}$

CASE 2: ODD POWER OF COSINE

Save one factor of cosine, then convert the rest to sine using the identity
$cos^2x = 1 - sin^2x$

Example 3:

odd

$$\int cos^3x \cdot sin^2x dx$$

$\int (\cos^2 x)(\sin^2 x)(\cos x)\,dx$

$\int (1-\sin^2 x)(\sin^2 x)(\cos x)\,dx$

$\int (\sin^2 x - \sin^4 x)(\cos x)\,dx$

$u = \sin x$
$du = \cos x\,dx$

$\int u^2 - u^4\,du$

$\frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$\boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$

CASE 3: EVEN POWERS OF SINE AND COSINE

Make repeated use of these identities:
$cos^2x = ½(1 + cos2x)$
$sin^2x = ½(1 - cos2x)$

Example 4:

$$\int cos^2x \cdot sin^2x dx$$

$\int \frac{1}{2}(1+\cos 2x) \cdot \frac{1}{2}(1-\cos 2x)\,dx$

$\frac{1}{4}\int (1+\cos 2x)(1-\cos 2x)\,dx$

$\frac{1}{4}\int 1 - \cos^2(2x)\,dx$

$\frac{1}{4}\int 1 - (\frac{1}{2}(1+\cos 4x))\,dx$

$\frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}\int \frac{1}{2} - \frac{1}{4}\int \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}(\frac{1}{2}x)$; $-\frac{1}{4}(\frac{1}{2})(\frac{1}{4})\int 4\cos 4x\,dx$

$u = 4x$
$du = 4dx$

$-\frac{1}{32}\int \cos u\,du$

$-\frac{1}{32}[\sin u]$

$\frac{1}{8}x$; $-\frac{1}{32}\sin 4x + C$

$\boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$

Example 5:

$$\int cos^4xdx$$

$\int (\cos^2 x)(\cos^2 x)\,dx$

$\int \frac{1}{2}(1+\cos 2x) \cdot \frac{1}{2}(1+\cos 2x)\,dx$

$\frac{1}{4}\int (1+\cos 2x)(1+\cos 2x)\,dx$

$\frac{1}{4}\int 1 + 2\cos 2x + \cos^2(2x)\,dx$

$\frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx$

$\frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}\int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}\int \frac{3}{2} + \frac{1}{4}\int 2\cos 2x + \frac{1}{4}\int \frac{1}{2}\cos 4x\,dx$

$\frac{1}{4}(\frac{3}{2}x)$; $\frac{1}{4}(\frac{2}{2})\int (\frac{2}{1})\cos 2x\,dx$; $\frac{1}{4}(\frac{1}{2})\int (\frac{4}{4})\cos 4x\,dx$

$u = 2x$
$du = 2dx$

$u = 4x$
$du = 4dx$

$\frac{3}{8}x$; $\frac{1}{4}\int \cos u\,du$; $\frac{1}{32}\int \cos u\,du$

$\frac{1}{4}\sin(2x)$; $\frac{1}{32}\sin(4x)$

$\boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$