AP Calculus BC
Unit 8 – Integration Techniques

Day 4 Notes: Mixed Integration

Example 1:

$$4\int \frac{4}{x^2+9}\,dx$$

$u = x$, $du = dx$, $a = 3$

$$= 4\int \frac{du}{u^2+a^2}$$

$$= 4\left(\tfrac{1}{3}\right)\arctan\left(\tfrac{x}{3}\right)+C$$

$$= \boxed{\tfrac{4}{3}\arctan\left(\tfrac{x}{3}\right)+C}$$

Example 2:

$$\tfrac{1}{2}\cdot 4\int \frac{2x}{x^2+9}\,dx$$

$u = x^2+9$, $du = 2x\,dx$

$$= 2\int \frac{du}{u}$$

$$= 2\ln|u|+C$$

$$= 2\ln|x^2+9|+C$$

$$= \boxed{2\ln(x^2+9)+C}$$

Example 3:

$$\int \frac{4x^2}{x^2+9}\,dx$$

Long Division

$$x^2+0x+9 \;\overline{)\;4x^2+0x+0}$$

$$-4x^2-0x-36$$

$$-36$$

$$\int 4 + \frac{-36}{x^2+9}\,dx$$

$$\int 4\,dx + \left|-36\int \frac{dx}{x^2+9}\right.$$

$u = x$, $du = dx$, $a = 3$

$$4x \qquad -36\int \frac{du}{u^2+a^2}$$

$$-36\left(\tfrac{1}{3}\right)\arctan\left(\tfrac{x}{3}\right)$$

$$-12\arctan\left(\tfrac{x}{3}\right)$$

$$\boxed{4x-12\arctan\left(\tfrac{x}{3}\right)+C}$$

Example 4:

$$\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$$

Separate Numerator

$$-\tfrac{1}{2}\int_0^1 \frac{-2x\,dx}{\sqrt{4-x^2}} + 3\int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx$$

$u = 4-x^2$, $du = -2x\,dx$

$u = x$, $du = dx$, $a = 2$

$$-\tfrac{1}{2}\int_4^3 \frac{du}{\sqrt{u}} \qquad 3\int \frac{du}{\sqrt{a^2-u^2}}$$

$$+\tfrac{1}{2}\int_3^4 u^{-1/2}\,du \qquad 3\arcsin\left(\tfrac{x}{2}\right)\Big]_0^1$$

$$+\tfrac{1}{2}\,\frac{u^{1/2}}{\frac{1}{2}}\Big]_3^4 \qquad 3\arcsin\left(\tfrac{1}{2}\right)-3\arcsin(0)$$

$$u^{1/2}\Big]_3^4 \qquad 3\left(\tfrac{\pi}{6}\right)-3(0)$$

$$\sqrt{4}-\sqrt{3} \qquad \tfrac{\pi}{2}-0$$

$$2-\sqrt{3} \qquad \boxed{2-\sqrt{3}+\tfrac{\pi}{2}}$$

Example 5:

$$\int (\cot x)[\ln(\sin x)]\,dx$$

$u = \ln(\sin x)$, $du = \frac{\cos x}{\sin x}\,dx = \cot x\,dx$

$$\int u\,du$$

$$\tfrac{1}{2}u^2+C$$

$$\boxed{\tfrac{1}{2}[\ln(\sin)]^2+C}$$

Example 6:

TRIG IDENTITIES

$$\tfrac{1}{2}\int \tan^2 2x\,dx$$

$u = 2x$, $du = 2dx$

$$\tfrac{1}{2}\int \tan^2(u)\,du$$

$$\tfrac{1}{2}\int \sec^2 u - 1\,du$$

$$\tfrac{1}{2}\int \sec^2 u\,du + \tfrac{1}{2}\int -1\,du$$

$$\tfrac{1}{2}(\tan u) + \tfrac{1}{2}(-u) + C$$

$$\tfrac{1}{2}\tan(2x) + \tfrac{1}{2}(-2x) + C$$

$$\boxed{\tfrac{1}{2}\tan(2x) - x + C}$$

Example 7:

$\frac{1}{3}\int \frac{3x^2}{\sqrt{16-x^6}}dx$

$u = x^3$

$du = 3x^2 dx$

$a = 4$

$\frac{1}{3}\int \frac{du}{\sqrt{a^2-u^2}}$

$\frac{1}{3}\arcsin\left(\frac{x^3}{4}\right)+C$

Example 8:

Add terms to numerator

$\int \frac{1+e^x-e^x}{1+e^x}dx$

$u = 1+e^x$

$du = e^x dx$

$\int \frac{1+e^x}{1+e^x}dx + \int \frac{-e^x}{1+e^x}dx$

$\int 1\,dx$ → x

$-\int \frac{du}{u}$ → $-\ln|u|$ → $-\ln|1+e^x|$

$x - \ln(1+e^x)+C$

**Procedures for Fitting Integrands to Basic Rules:

1) **Expand (numerator).** *Example:* $(1+e^x)^2 = 1 + 2e^x + e^{2x}$

2) **Separate numerator.** *Example:* $\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$

3) **Complete the square.** *Example:* $\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$

4) **Divide improper rational function.** *Example:* $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$

5) **Add and subtract terms in numerator.**
Example: $\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$

6) **Use trigonometric identities.** *Example:* $\cot^2 x = \csc^2 x - 1$

7) **Multiply and divide by Pythagorean conjugate.**
Example: $\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$