AP Calculus BC
Unit 8 – Integration Techniques

Day 3 Notes: Inverse Trig Functions – Integration

Let $\boldsymbol{u}$ be a function of $\boldsymbol{x}$ and let $\boldsymbol{a}$ be a constant. Then

$$\int \frac{du}{\sqrt{a^2 - u^2}} = arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} arcsec\frac{|u|}{a} + C$$

Example 1:

$$\int \frac{dx}{4 + x^2}$$

$u = x$
$du = dx$
$a = 2$

$$\int \frac{du}{a^2 + u^2}$$

$$\frac{1}{a} \arctan \frac{u}{a} + C$$

$$\boxed{\frac{1}{2} \arctan \frac{x}{2} + C}$$

Example 2:

$$\int \frac{dx}{\sqrt{9 - 4x^2}}$$

$u = 2x$
$du = 2dx$
$\frac{1}{2}du = dx$
$a = 3$

$$\int \frac{\frac{1}{2}du}{\sqrt{a^2 - u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \frac{1}{2} \arcsin \frac{u}{a} + C$$

$$= \boxed{\frac{1}{2} \arcsin \frac{2x}{3} + C}$$

Example 3:

$$\int \frac{dx}{\sqrt{e^{2x} - 1}}$$

$u = e^x$
$du = e^x dx$
$\frac{1}{e^x} du = dx$
$a = 1$

$$\int \frac{\frac{1}{e^x} du}{\sqrt{u^2 - a^2}} = \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

$$= \frac{1}{1} \operatorname{arcsec} \frac{|e^x|}{1} + C$$

$$\boxed{\operatorname{arcsec}(e^x) + C}$$

Example 4:
(Complete the square..)

$$\int \frac{dx}{x^2 + 6x + 13}$$

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 + 13 - \left(\frac{6}{2}\right)^2$$

$$(x+3)^2 + 4$$

$u = x + 3$
$du = dx$
$a = 2$

$$\int \frac{dx}{(x+3)^2 + 4}$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$= \boxed{\frac{1}{2} \arctan \frac{x+3}{2} + C}$$

Example 5:
(Complete the square & split the integral into 2…)

$$\int \frac{2x-5}{x^2+2x+2}\,dx$$

$u = x^2+2x+2$
$du = 2x+2\,dx$

$$\int \frac{2x+2-2-5}{x^2+2x+2} = \int \frac{2x+2\,dx}{x^2+2x+2} + \int \frac{-7}{x^2+2x+2}\,dx$$

$u = x+1$
$du = dx$
$a = 1$

$$= \int \frac{1}{u}\,du + \int \frac{-7}{(x+1)^2+1}$$

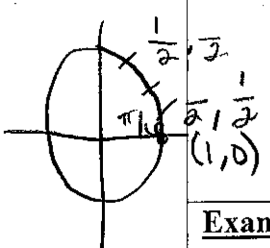
$x^2+2x+\left(\frac{2}{2}\right)^2+2-\left(\frac{2}{2}\right)^2$
$(x+1)^2+1$

$$= \ln|u| + \int \frac{-7}{u^2+a^2}$$

$$= \boxed{\ln|x^2+2x+2| - 7\arctan(x+1)+C}$$

Example 6:
(Simplify the integrand first…)

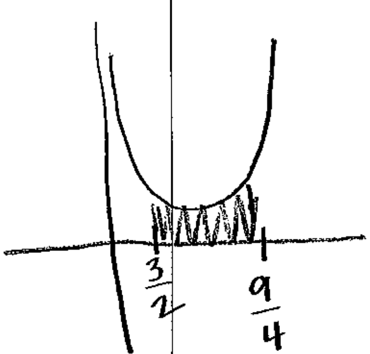
$$\int \frac{\mathbf{x^4-1}}{\mathbf{x^2+1}}\,\mathbf{dx} = \int \frac{(x^2-1)\cancel{(x^2+1)}}{\cancel{(x^2+1)}}\,dx$$



$$\int (x^2-1)\,dx$$

$$\frac{x^3}{3} - \frac{x}{1} + C = \boxed{\frac{1}{3}x^3 - x + C}$$

Example 7:
Find the area of the region bounded by the graph of $f(x) = \frac{1}{\sqrt{3x-x^2}}$, the x-axis, and the lines x = 3/2 and x = 9/4.



$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x-x^2}}$$

$3x-x^2 = -x^2+3x$
$= -(x^2-3x)$
$-\left(x^2-3x+\left(\frac{-3}{2}\right)^2-\left(\frac{-3}{2}\right)^2\right)$
$-\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]$

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$u = x-\frac{3}{2}$
$du = dx$
$a = 3/2$

$$\int \frac{1}{\sqrt{a^2-u^2}}$$

$\frac{9}{4}-\frac{6}{4}$

$$\arcsin \frac{x-\frac{3}{2}}{3/2}\Bigg]_{3/2}^{9/4} = \arcsin\left(\frac{\frac{9}{4}-\frac{3}{2}}{\frac{3}{2}}\right) - \arcsin\left(\frac{\frac{3}{2}-\frac{3}{2}}{\frac{3}{2}}\right)$$

$$= \arcsin\frac{1}{2} - \arcsin 0$$

$$\frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$$