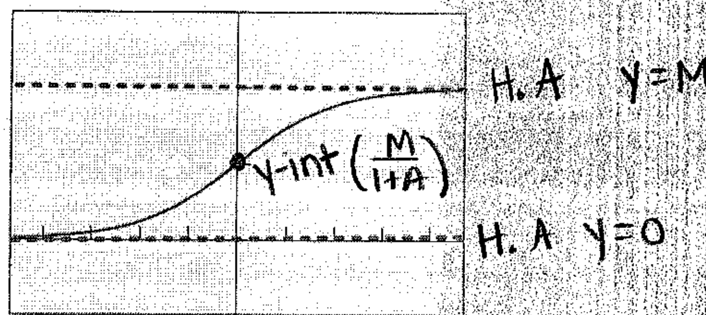
AP Calculus BC
Unit 8 – Integration Techniques

Day 10 Notes: Logistics Differential Equations

General Logistic Formula:

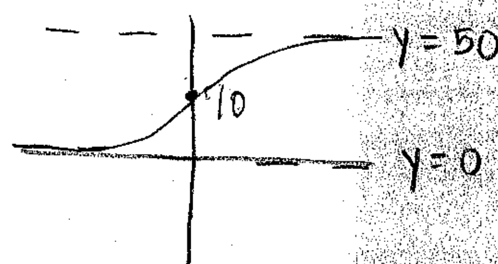
$$P = \frac{M}{1 + Ae^{-(Mk)t}}$$



M = carrying capacity
k = growth constant

Example 1:

Let $f(x) = \frac{50}{1+4e^{-0.2x}}$



a) Find where f(x) is continuous. $\boxed{(-\infty, \infty)}$

b) Find $\lim_{x\to\infty} f(x)$. $\boxed{50} = M$

c) Find $\lim_{x\to-\infty} f(x)$. $\boxed{0}$

d) Find the y-intercept of the graph of f(x). $\frac{50}{1+4} = \frac{50}{5} = \boxed{10}$

e) Find all horizontal asymptotes of the graph of f(x). $\boxed{y=50 \text{ \& } y=0}$

f) Find the carrying capacity of f(x). $\boxed{50}$

Logistics Differential Equation:

$$\frac{dP}{dt} = kP(M - P)$$

Things to remember about logistics differential equations:

1) *M = carrying capacity*
2) *No matter what the initial population is, $\lim_{t\to\infty} P(t) = M$.*
3) *The growth rate of $P(t)$ is increasing fastest when $P = M/2$.*
4) *You can solve a logistics differential equation by separating the variables and using partial fractions.*

Example 2: The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $dP/dt = 0.008P(100 - P)$, where t is measured in years.

a) What is the carrying capacity for bears in this wildlife preserve? $\boxed{100 \text{ bears}}$

b) What is the bear population when the population is growing the fastest? $\frac{100}{2} = \boxed{50 \text{ bears}}$

c) What is the rate of change of the population when it is growing the fastest?

$\frac{dP}{dt} = .008(50)(100-50)$
$= \boxed{20 \text{ bears}}$

d) If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$? $\boxed{100} = M$

Doesn't matter

Example 3: In 1985 and 1987, the Michigan Department of Natural Resources airlifted 61 moose from Algonquin Park, Ontario to Marquette County in the Upper Peninsula. It was originally hoped that the population P would reach carrying capacity in about 25 years with a growth rate of

$$\frac{dP}{dt} = 0.0003P(1000 - P)$$

a) According to the model, what is the carrying capacity? $\boxed{1000 \text{ moose}}$

b) Solve the differential equation with the initial condition P(0) = 61.

$$\int \frac{dP}{P(1000-P)} = \int .0003\,dt$$

$$\frac{1}{P(1000-P)} = \frac{A}{P}\,(1000-P) + \frac{B}{1000-P}\,(P)$$

$1 = A(1000-P) + B(P)$

$\underline{P=1000}$: $1 = A(\cancel{1000}-1000) + B(1000)$

$1 = B(1000)$

$B = .001$

$\underline{P=0}$: $1 = A(1000-0) + \cancel{B(0)}$

$1 = A(1000)$

$A = .001$

$$\int \frac{.001}{P} + \frac{.001}{1000-P} = \int .0003\,dt$$

$.001\ln|P| - .001\ln|1000-P| = .0003t + C$

$.001(\ln|P| - \ln|1000-P|) = .0003t + C$

$\ln|P| - \ln|1000-P| = .3t + C$

$\ln|1000-P| - \ln|P| = -.3t - C$

$\ln\left(\frac{1000-P}{P}\right) = -.3t - C$

$\frac{1000-P}{P} = e^{-.3t-C}$

$\frac{1000}{P} - 1 = e^{-.3t-C}$

$\frac{1000}{P} = 1 + e^{-.3t}e^{-C}$

$P(0) = 61$

$\frac{1000}{61} = 1 + e^{-.3(0)}e^{-C}$

$16.393 = 1 + e^{-C}$

$15.393 = e^{-C}$

$\frac{1000}{P} = 1 + 15.393e^{-.03t}$

$P = \frac{1000}{1 + 15.393e^{-.03t}}$

Example 4: A 2000-gallon tank contains guppies. The rate of growth of the guppies in the tank is $\frac{dP}{dt} = P(0.225 - 0.0015P)$, where t is in weeks.

a) Without actually solving the differential equation, what is $\lim_{t \to \infty} P(t)$? Interpret the limit in the context of the problem.

$\frac{dP}{dt} = .0015P(150-P)$

$\boxed{150}$

The most guppies that will be in the tank is 150.

b) Solve the differential equation, given that P(0) = 20.

$\frac{dP}{dt} = .0015P(150-P)$

$\int \frac{dP}{P(150-P)} = \int .0015\,dt$

$\frac{1}{P(150-P)} = \frac{A}{P}(150-P) + \frac{B}{150-P}(P)$

$1 = A(150-P) + B(P)$

P=150: $1 = A(\cancel{150-150}) + B(150)$

$1 = 150B$

$B = \frac{1}{150}$

P=0: $1 = A(150-0) + \cancel{B(0)}$

$1 = A(150)$

$A = \frac{1}{150}$

$\int \frac{1/150}{P} + \frac{\frac{1}{150}}{150-P}\,dP = \int .0015\,dt$

$\frac{1}{150}\ln|P| - \frac{1}{150}\ln|150-P| = .0015t$

$\frac{1}{150}(\ln P - \ln(150-P)) = .0015t + C$

$\ln P - \ln(150-P)) = .225t + C$

$\ln(150-P) - \ln P = -.225t - C$

$\ln\left(\frac{150-P}{P}\right) = -.225t - C$

$\frac{150-P}{P} = e^{-.225t-C}$

$\frac{150}{P} - 1 = e^{-.225t}e^{-C}$

$\frac{150}{P} = 1 + e^{-.225t}e^{-C}$

$P(0) = 20$

$\frac{150}{20} = 1 + \cancel{e^{-.225(0)}}e^{-C}$

$7.5 = 1 + e^{-C}$

$6.5 = e^{-C}$

$\frac{150}{P} = 1 + 6.5e^{-.225t}$

$$\boxed{P = \frac{150}{1 + 6.5e^{-.225t}}}$$