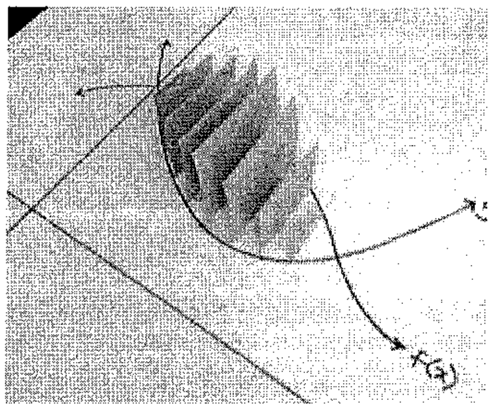
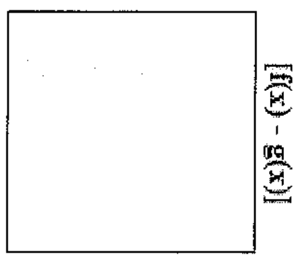
AP Calculus
Unit 7 – Advanced Integration & Applications

Day 7 Notes: Volumes of Solids with Known Cross Sections

Cross Sections that are Squares





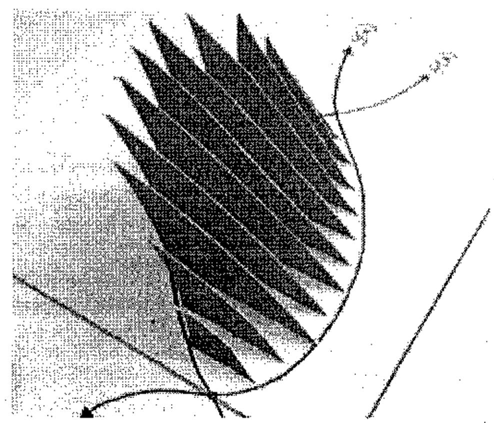
Find the area of the square above in terms of $f(x)$ and $g(x)$.

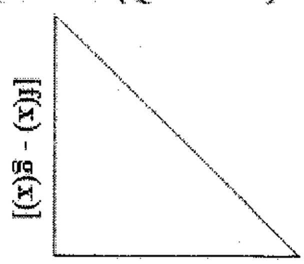
Area $= [f(x)-g(x)]^2$

If all of the cross sectional areas were added up, the total would be the volume of the solid pictured. How do we write this in calculus using an integral?

Volume $= \int_a^b [f(x)-g(x)]^2\,dx$

Cross Sections that are Isosceles Right Triangles





Find the area of the triangle above in terms of $f(x)$ and $g(x)$.

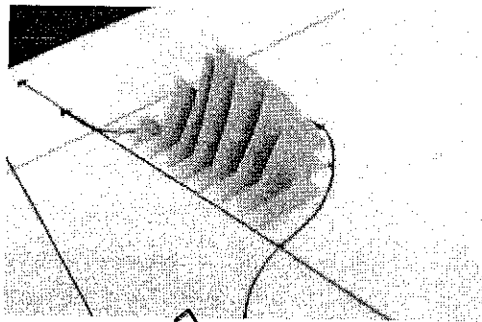
Area $= \frac{1}{2}[f(x)-g(x)]^2$

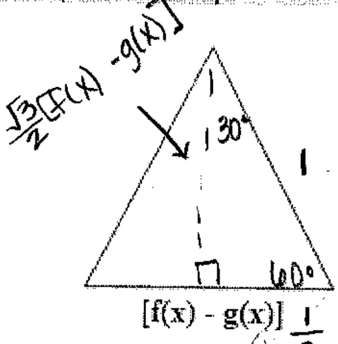
If all of the cross sectional areas were added up, the total would be the volume of the solid pictured. How do we write this in calculus using an integral?

Volume $= \int_a^b \frac{1}{2}[f(x)-g(x)]^2\,dx$

or $\frac{1}{2}\int_a^b [f(x)-g(x)]^2\,dx$

Cross Sections that are Equilateral Triangles





$\left(\frac{1}{2}\right)^2 + x^2 = 1^2$

$\frac{1}{4} + x^2 = 1$

$x^2 = \frac{3}{4}$

$x = \frac{\sqrt{3}}{2}$

Find the area of the triangle above in terms of $f(x)$ and $g(x)$.

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}[f(x)-g(x)] \cdot \frac{\sqrt{3}}{2}[f(x)-g(x)]$$

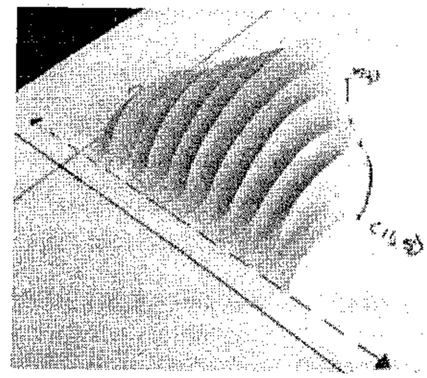
$$A = \frac{\sqrt{3}}{4}[f(x)-g(x)]^2$$

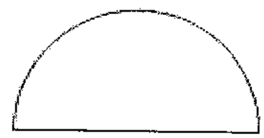
If all of the cross sectional areas were added up, the total would be the volume of the solid pictured. How do we write this in calculus using an integral?

$$\text{Volume} = \int_a^b \frac{\sqrt{3}}{4}[f(x)-g(x)]^2\,dx$$

$$\text{or } \frac{\sqrt{3}}{4}\int_a^b [f(x)-g(x)]^2\,dx$$

Cross Sections that are Semicircles





Find the area of the semicircle above in terms of $f(x)$ and $g(x)$.

$$A = \frac{1}{2}\pi r^2$$

$$= \frac{1}{2}\pi\left[\frac{1}{2}[f(x)-g(x)]\right]^2$$

$$= \frac{1}{2}\pi\left[\frac{1}{4}[f(x)-g(x)]^2\right]$$

$$= \frac{1}{8}\pi[f(x)-g(x)]^2$$

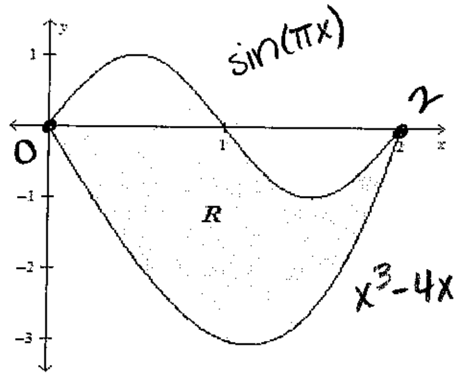
If all of the cross sectional areas were added up, the total would be the volume of the solid pictured. How do we write this in calculus using an integral?

$$\text{Vol} = \int_a^b \frac{1}{8}\pi[f(x)-g(x)]^2\,dx$$

$$\text{or } \frac{1}{8}\pi\int_a^b [f(x)-g(x)]^2\,dx$$

What do you notice about the integral-defined formulas for finding the volume of solids with certain cross sections?

Region R is bounded by $y = \sin(\pi x)$ and $y = x^3 - 4x$.



Find the volume of the solids formed whose cross sections are the shapes indicated below. The cross sections are perpendicular to the x – axis.

a. Cross sections are equilateral triangles

$$V = \frac{\sqrt{3}}{4}\int_a^b [f(x) - g(x)]^2\,dx$$

$$V = \frac{\sqrt{3}}{4}\int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{4.321}$$

b. Cross sections are semi-circles

$$V = \frac{1}{8}\pi\int_a^b [f(x) - g(x)]^2\,dx$$

$$= \frac{1}{8}\pi\int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{3.918}$$

c. Cross sections are isosceles right triangles

$$V = \frac{1}{2}\int_a^b [f(x) - g(x)]^2\,dx$$

$$= \frac{1}{2}\int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{4.989}$$

d. Cross sections are squares.

$$V = \int_a^b [f(x) - g(x)]^2\,dx$$

$$= \int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{9.978}$$

e. Cross sections are rectangles whose height is twice the length of the base.

$2[f(x)-g(x)]$

$[f(x)-g(x)]$

$$A = 2[f(x) - g(x)]^2$$

$$V = 2\int_a^b [f(x) - g(x)]^2$$

$$V = 2\int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{19.957}$$

f. Cross sections are rectangles whose height is one-third the length of the base.

$$A = \frac{1}{3}[f(x) - g(x)]^2$$

$\frac{1}{3}[f(x)-g(x)]$

$[f(x)-g(x)]$

$$V = \frac{1}{3}\int_a^b [f(x) - g(x)]^2\,dx$$

$$= \frac{1}{3}\int_0^2 [\sin(\pi x) - (x^3-4x)]^2\,dx$$

$$= \boxed{3.326}$$

CALCULATOR PERMITTED

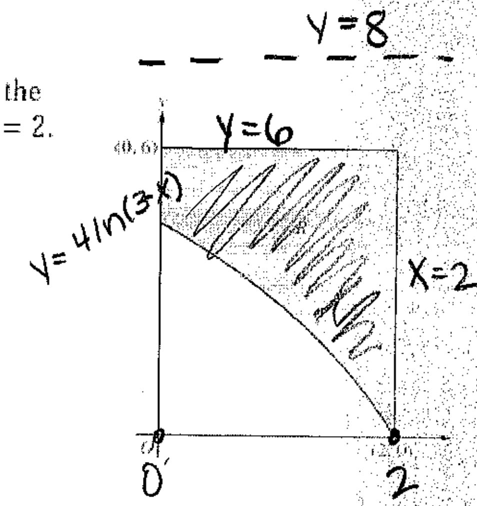
2010 AP® CALCULUS AB (Form B)
Question 1

In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a $\boxed{\text{square}}$. Find the volume of the solid.



(a) $\int_0^2 6-[4\ln(3-x)]\,dx = \boxed{6.817}$

(b) $V = \pi\int_0^2 [4\ln(3-x) - 8]^2 - [6-8]^2\,dx = \boxed{168.180}$

(c) $V = \int_a^b [f(x) - g(x)]^2\,dx$

$V = \int_0^2 [6 - (4\ln(3-x))]^2\,dx = \boxed{26.267}$