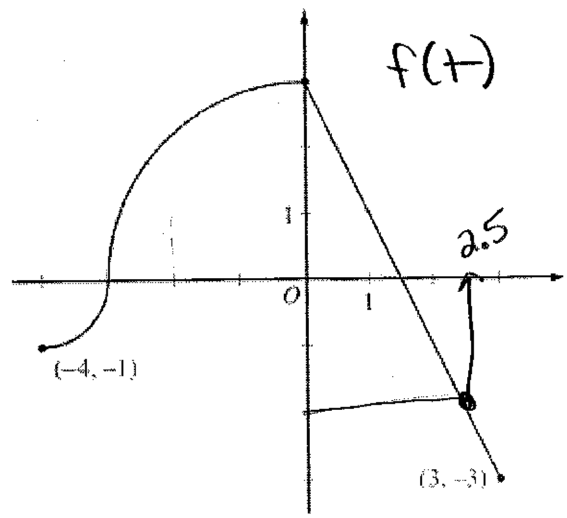
AP Calculus AB
Unit 7 – Day 4 – Warm-up

Name: Answer Key *

2011 AP® CALCULUS AB
Question 4



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) \rightarrow \quad g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt$

$$= -6 + -\int_{-3}^{0} f(t)\,dt \quad \underline{\text{area}}$$

$$= -6 + -\tfrac{1}{4}\pi(3)^2$$

$$= \boxed{-6 + \tfrac{9}{4}\pi}$$

$g'(x) \rightarrow \quad g(x) = 2x + \int_0^{x} f(t)\,dt$

$$g'(x) = 2 + f(x)(1)$$

$$g'(x) = 2 + f(x)$$

$$g'(-3) = 2 + f(-3)$$

$$= 2 + 0$$

$$= \boxed{2}$$

(b) Extreme Value Thm.

$g(x) = 2x + \int_0^x f(t)\,dt$

$g'(x) = 2 + f(x)$

$2 + f(x) = 0$

$f(x) = -2$

look@ graph $x = 2.5$

$g'(x)=0$

$g(2.5) = 2(2.5) + \int_0^{2.5} f(t)\,dt = 5 + \frac{1}{2}(1.5)(3) - \frac{1}{2}(1)(2)$
$= 5 + 2.25 - 1 = 6.25$

lower #

$g(-4) = 2(-4) + \int_0^{-4} f(t)\,dt$

$2(-4) + -\int_{-4}^{0} f(t)\,dt = -8 - \left(\frac{1}{4}\pi(3)^2 - \frac{1}{4}\pi(1)^2\right)$

$= -8 - \left(\frac{9}{4}\pi - \frac{1}{4}\pi\right)$

$= -8 - \left(\frac{8}{4}\pi\right)$

$= -8 - 2\pi$

highest #

$g(3) = 2(3) + \int_0^3 f(x)\,dx = 6 + \frac{1}{2}(1.5)(3) - \frac{1}{2}(1.5)(3) = 6$

$\boxed{\text{The E.V.T guarantees the absolute maximum of } g(x) \text{ to occur at } x = 2.5}$

(c) $g(x)$ has a point of inflection when $g''(x)$ changes signs.

$g'(x) = 2 + f(x)$

$g''(x) = f'(x)$

If $g''(x) = f'(x)$ changes signs, then $f(x)$ has a rel. max or rel. min. $\boxed{x = 0}$

(d) avg rate of change of f(x) on $-4 \le x \le 3$

$$\frac{f(-4)-f(3)}{-4-3} = \frac{-1-(-3)}{-7} = \boxed{\frac{2}{-7}}$$

There is no such value of c such that $f'(c) = -2/7$ b/c the M.V.T is not applicable since f(x) is not differentiable for all values on $-4 < x < 3$. (cusp at $x=0$).

AP Calculus
Unit 7 – Advanced Integration & Applications

Day 4 Notes: Slope Fields

A **slope field** is a pictorial representation of all of the possible solutions to a given differential equation.

Remember that a differential equation is the first derivative of a function, $f'(x)$ or $\frac{dy}{dx}$. Thus, the solution to a differential equation is the function, $f(x)$ or y.

There is an infinite number of solutions to the differential equation $\frac{dy}{dx} = x - 1$. Show your work and explain why.

$\frac{dy}{dx} = x - 1$

$\int dy = \int (x-1)\,dx$

$y = \frac{1}{2}x^2 - x + C$ ← General soln. of $\frac{dy}{dx}$

Since C can be any #, there are an infinite number of solutions.

Given the differential equation below, compute the slope for each point
Indicated on the grid to the right.

Then, make a small mark that approximates the slope through the point.

$$\frac{dy}{dx} = x - 1$$

SLOPE FUNCTION

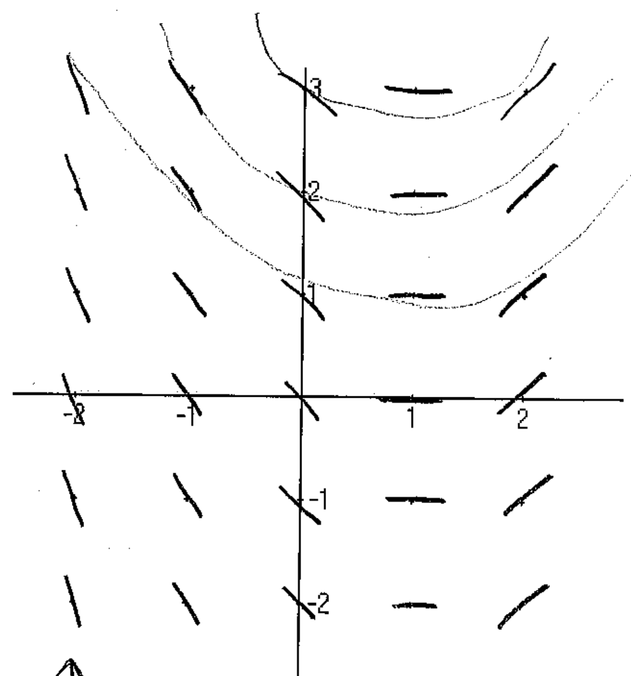
$\frac{dy}{dx}$ @ $(-2,3) = -2-1 = -3$

$\frac{dy}{dx}$ @ $(-1,3) = -1-1 = -2$

$\frac{dy}{dx}$ @ $(0,3) = 0-1 = -1$

$\frac{dy}{dx}$ @ $(1,3) = 1-1 = 0$

$\frac{dy}{dx}$ @ $(2,3) = 2-1 = 1$



same b/c y value doesn't change the slope

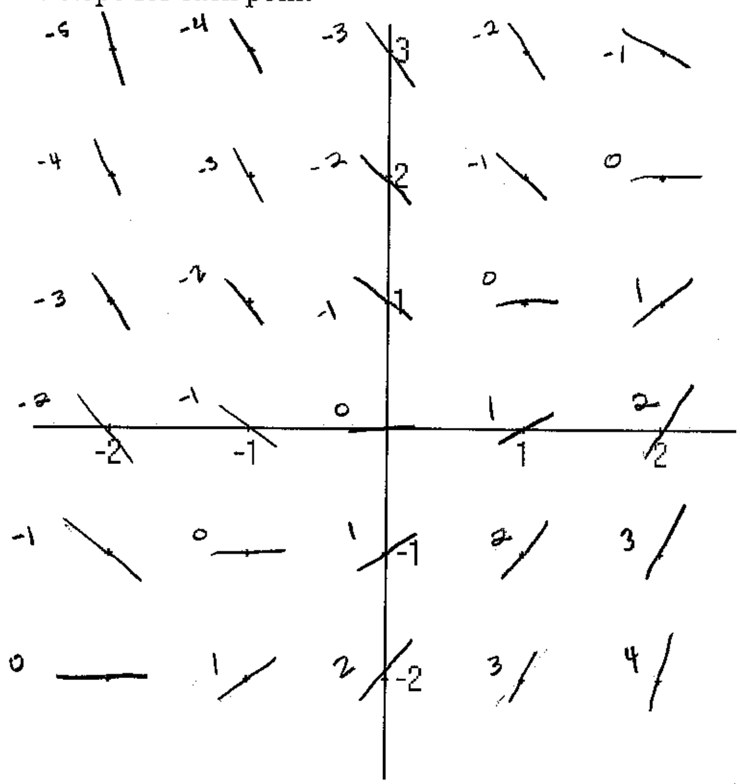
Notice how the segments drawn on the grid above would form parabolas if they were connected.

Given the differential equation below, compute the slope for each point Indicated on the grid to the right.

Then, make a small mark that approximates the slope through the point.

$$\frac{dy}{dx} = x - y$$

Slope eqn.



$\frac{dy}{dx} @ (-2,3) = -2-3 = -5$

$\frac{dy}{dx} @ (-2,2) = -2-2 = -4$

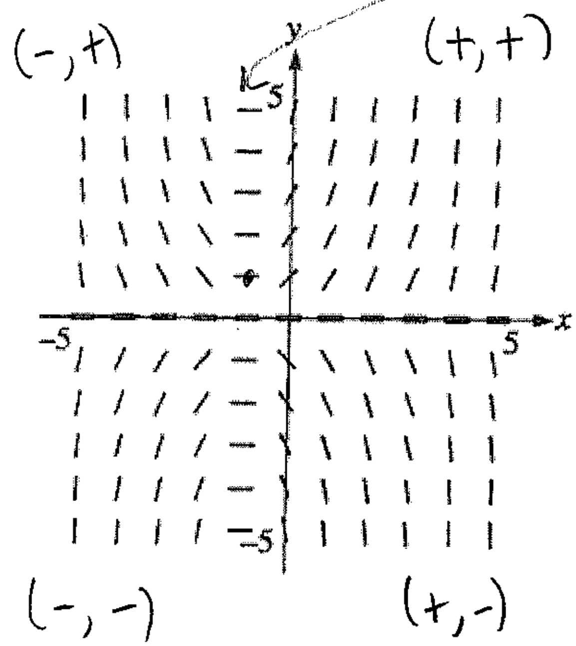
$\frac{dy}{dx} @ (-2,1) = -2-1 = -3$

$\frac{dy}{dx} @ (-2,0) = -2-0 = -2$

$\frac{dy}{dx} @ (-2,-1) = -2-(-1) = -1$

$\frac{dy}{dx} @ (-2,-2) = -2-(-2) = 0$

Shown below is a slope field for which of the following differential equations? Explain your reasoning for each of the choices below.



$(-1,1)$

Slope

~~(A)~~ $\frac{dy}{dx} = xy$ $\quad (-1)(-1) = 1$

~~(B)~~ $\frac{dy}{dx} = xy - y$ $\quad y(x-1) = 0$ $\quad 1(-1-1) = -2$

(C) $\frac{dy}{dx} = xy + y$ $\quad y(x+1)$ $\quad 1(-1+1) = 0$

~~(D)~~ $\frac{dy}{dx} = xy + x$ $\quad x(y+1)$ $\quad -1(1+1) = -2$

~~(E)~~ $\frac{dy}{dx} = (x+1)^3$

$(-1+1)^3 = 0$

$(-2+1)^3 = -1$

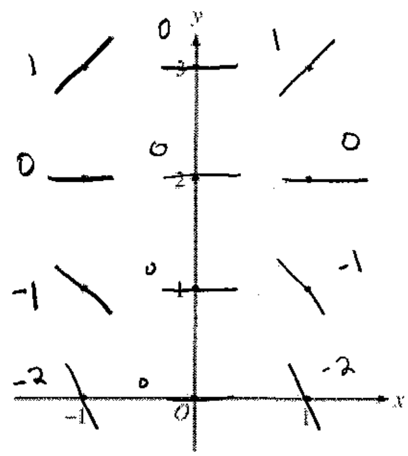
2004 AP® CALCULUS AB
Question 5 (Form B)

Consider the differential equation $\frac{dy}{dx} = x^4(y-2)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



(a) $\frac{dy}{dx} = \underbrace{x^4(y-2)}_{\text{slope equation}}$

(b) The slope will be negative at any point (x, y) such that $y<2$ but $x \neq 0$.

$x^4(y-2)$

always ⊕ neg when $y<2$

(c) $\frac{dy}{dx} = x^4(y-2)$

$\frac{dy}{y-2} = \frac{x^4\cancel{(y-2)}\,dx}{\cancel{y-2}}$

$\int (y-2)^{-1}\,dy = \int x^4\,dx$

$\frac{(y-2)^0}{0} = \frac{1}{5}x^5 + C$

$\ln|y-2| = \frac{1}{5}x^5 + C$

$f(0) = 0$ (x, y)

$\ln|0-2| = \frac{1}{5}(0)^2 + C$

$\ln 2 = 0 + C$

$C = \ln 2$

$\cancel{e^{\ln}}|y-2| = e^{\frac{1}{5}x^5 + \ln 2}$

$|y-2| = e^{\frac{1}{5}x^5} + \ln 2$

$|y-2| = e^{\frac{1}{5}x^5} \cdot \cancel{e^{\ln 2}}$

$|y-2| = 2e^{\frac{1}{5}x^5}$

$y-2 = 2e^{1/5x^5}$ or $y-2 = -2e^{\frac{1}{5}x^5}$

$y = \cancel{2+}2e^{\frac{1}{5}x^5}$ or $y = 2-2e^{\frac{1}{5}x^5}$

$\boxed{f(x) = 2-2e^{\frac{1}{5}x^5}}$

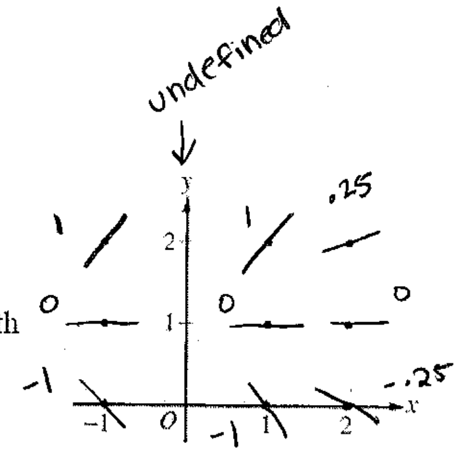
2008 AP® CALCULUS AB
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \to \infty} f(x)$.



ⓐ $\frac{dy}{dx} = \frac{y-1}{x^2}$

slope eqn

ⓑ $\frac{dy}{dx} = \frac{y-1}{x^2}$

$\frac{\cancel{x^2}dy}{\cancel{x^2}} = \frac{(y-1)dx}{x^2}$

$\frac{dy}{y-1} = \frac{\cancel{(y-1)}x^{-2}dx}{\cancel{y-1}}$

$\int (y-1)^{-1}dy = \int x^{-2}dx$

$\frac{(y-1)^0}{0} = -x^{-1} + C$

$\ln|y-1| = -\frac{1}{x} + C$

$f(2) = 0$
$\ \ x \quad\ y$

$\ln|0-1| = -\frac{1}{2} + C$

$\ln 1 = -\frac{1}{2} + C$

$0 = -\frac{1}{2} + C$

$C = 1/2$

$\cancel{e}^{\cancel{\ln}|y-1|} = e^{-\frac{1}{x}+\frac{1}{2}}$

$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}}$

$y-1 = e^{-\frac{1}{x}+\frac{1}{2}}$ or $y-1 = -e^{-\frac{1}{x}+\frac{1}{2}}$

$\cancel{y = 1+e^{-\frac{1}{x}+\frac{1}{2}}}$ or $y = 1-e^{-\frac{1}{x}+\frac{1}{2}}$

$f(2) = 0$ $\quad 1-e^{-\frac{1}{2}+\frac{1}{2}}$

$\boxed{f(x) = 1-e^{-\frac{1}{x}+\frac{1}{2}}}$

ⓒ $\lim_{x\to\infty} 1-e^{-\frac{1}{x}+\frac{1}{2}}$

$= 1 - e^{\frac{-1}{\infty}+\frac{1}{2}}$

$= 1 - e^{1/2}$

$= \boxed{1-\sqrt{e}}$