AP Calculus
Unit 7 – Advanced Integration & Applications

Day 3 Notes: Solving Differential Equations

Given below are differential equations with given initial condition values. Find the particular solution, $y = f(x)$, for each differential equation that satisfies the given initial condition.

1. $\frac{dy}{dx} = 6x^2 + 6x + 2$ and $f(-1) = 2$

$\int dy = \int 6x^2 + 6x + 2\, dx$

$y = 2x^3 + 3x^2 + 2x + C$

$2 = 2(-1)^3 + 3(-1)^2 + 2(-1) + C$

$2 = -2 + 3 - 2 + C$

$2 = -1 + C$

$C = 3$

$\boxed{f(x) = 2x^3 + 3x^2 + 2x + 3}$

2. $\frac{dy}{dx} = \frac{1 + 12x^{3/2}}{2\sqrt{x}}$ ($2x^{1/2}$) and $f(0) = 2$

$\int dy = \int \frac{1}{2}x^{-1/2} + 6x\, dx$ $\quad \frac{1}{2} \cdot \frac{2}{1}$

$y = \frac{\frac{1}{2}x^{1/2}}{1/2} + \frac{6x^2}{2} + C$

$y = x^{1/2} + 3x^2 + C$

$2 = (0)^{1/2} + 3(0)^2 + C$

$C = 2$

$\boxed{f(x) = x^{1/2} + 3x^2 + 2}$

3. $\frac{dy}{dx} = \frac{x^2 + 2x}{2y}$ and $f(0) = 2$ $(+)$

$\int 2y\, dy = \int x^2 + 2x\, dx$

$y^2 = \frac{1}{3}x^3 + x^2 + C$

$2^2 = \frac{1}{3}(0)^3 + (0)^2 + C$

$C = 4$

$\sqrt{y^2} = \pm\sqrt{\frac{1}{3}x^3 + x^2 + 4}$

$y = \pm\sqrt{\frac{1}{3}x^3 + x^2 + 4}$

$\boxed{f(x) = \sqrt{\frac{1}{3}x^3 + x^2 + 4}}$

4. $\frac{dy}{dx} = \frac{x + 2}{y}$ and $f(1) = -3$ $(-)$

$\int y\, dy = \int x + 2\, dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2x + C$

$\frac{1}{2}(-3)^2 = \frac{1}{2}(1)^2 + 2(1) + C$

$\frac{9}{2} = \frac{1}{2} + 2 + C$

$\frac{9}{2} = \frac{5}{2} + C$

$2 = C$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2x + 2$

$y^2 = x^2 + 4x + 4$

$y = \pm\sqrt{x^2 + 4x + 4}$

$\boxed{f(x) = -\sqrt{x^2 + 4x + 4}}$

5. $\frac{dy}{dx} = x^4(y-2)$ and $f(0) = 0$

$x \quad y$

$\frac{dy}{y-2} = \frac{x^4(y-\cancel{2})\,dx}{y\cancel{-2}}$

$\int (y-2)^{-1}\,dy = \int x^4\,dx$

$\frac{(y-2)^0}{0} = \frac{1}{5}x^5 + C$

$\boxed{\ln|y-2| = \frac{1}{5}x^5 + C}$

$\ln|0-2| = \frac{1}{5}(0)^5 + C$

$\ln 2 = C$

$\cancel{e}^{\cancel{\ln}|y-2|} = e^{\frac{1}{5}x^5 + \ln 2}$

$|y-2| = e^{1/5x^5 + \ln 2}$

$|y-2| = e^{\frac{1}{5}x^5} \cdot \cancel{e^{\ln 2}}$

$|y-2| = e^{\frac{1}{5}x^5} \cdot 2$

$|y-2| = 2e^{\frac{1}{5}x^5}$

$y-2 = 2e^{\frac{1}{5}x^5}$ or $y-2 = -2e^{\frac{1}{5}x^5}$

$\cancel{y = 2 + 2e^{\frac{1}{5}x^5}}$ or $y = 2 - 2e^{\frac{1}{5}x^5}$

b/c $f(0) = 0$

$\boxed{f(x) = 2 - 2e^{\frac{1}{5}x^5}}$

6. $\frac{dy}{dx} = \frac{y-1}{x^2}$ and $f(2) = 0$

$x \quad y$

$\frac{\cancel{x^2}\,dy}{\cancel{x^2}} = \frac{(y-1)\,dx}{x^2}$

$\frac{dy}{y-1} = \frac{(y\cancel{-1})x^{-2}\,dx}{y\cancel{-1}}$

$\int (y-1)^{-1}\,dy = \int x^{-2}\,dx$

$\frac{(y-1)^0}{0} = \frac{x^{-1}}{-1} + C$

$\boxed{\ln|y-1| = -\frac{1}{x} + C}$

$\ln|0-1| = -\frac{1}{2} + C$

$\ln 1 = -\frac{1}{2} + C$

$0 = -1/2 + C$

$C = 1/2$

$\cancel{e}^{\cancel{\ln}|y-1|} = e^{-\frac{1}{x} + \frac{1}{2}}$

$|y-1| = e^{-\frac{1}{x} + \frac{1}{2}}$

$y-1 = e^{-1/x + 1/2}$ or $y-1 = -e^{-1/x + 1/2}$

$y = \cancel{1 + e^{-\frac{1}{x} + \frac{1}{2}}}$ or $y = 1 - e^{-\frac{1}{x} + \frac{1}{2}}$

$-1 + e^{-\frac{1}{2} + \frac{1}{2}}$ $\qquad$ $1 - e^{-\frac{1}{2} + \frac{1}{2}}$

$1 + 1$ $\qquad$ $1 - 1$

$\boxed{f(x) = 1 - e^{-\frac{1}{x} + \frac{1}{2}}}$

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Question 6

$$\frac{dy}{dx} = \frac{3x^2}{e^y}$$

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^y}$

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

$x \quad y$

(b) Find the domain and range of the function f found in part (a).

(a) $\frac{dy}{dx} \times \frac{3x^2}{e^y}$

$\int e^y\,dy = \int 3x^2\,dx$

$\boxed{e^y = x^3 + c}$

$e^{1/2} = (0)^3 + c$

$e^{1/2} = c$

$\cancel{\ln}(\cancel{e}^y) = \ln(x^3 + e^{1/2})$

$\boxed{y = \ln(x^3 + e^{1/2})}$

or

$\boxed{y = \ln(x^3 + \sqrt{e})}$

(b) Domain:

$x^3 + \sqrt{e} > 0$

$x^3 > -\sqrt{e}$

$x^3 > -(e^{1/2})$

$x > -(e^{1/2})^{1/3}$

$x > -e^{1/6}$

$x > -\sqrt[6]{e}$

$\boxed{(-\sqrt[6]{e}, \infty)}$

$\boxed{\text{Range: } (-\infty, \infty)}$