AP Calculus AB
Unit 7 – Day 3 – Assignment

Name: Answer Key*

AP® CALCULUS AB — 2001 Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

$\frac{dy}{dx} = (y^2)(6-2x)$

(a) Find 2nd derivative

Product Rule

$$\frac{d^2y}{dx^2} = (2y\frac{dy}{dx})(6-2x) + (y^2)(-2)$$

plug in $y^2(6-2x)$

$$= 2y[y^2(6-2x)](6-2x) + (y^2)(-2)$$

Plug in $x=3$ & $y=1/4$

$$= 2\left(\tfrac{1}{4}\right)\left[\left(\tfrac{1}{4}\right)^2(6-2(3))\right](6-2(3)) + \left(\tfrac{1}{4}\right)^2(-2)$$

$$= \tfrac{1}{2}\left[\tfrac{1}{16}(0)\right](0) + \tfrac{1}{16}(-2)$$

$$= 0 - \tfrac{1}{8}$$

$$= \boxed{-1/8}$$

(b)

$$\frac{dy}{dx} = y^2(6-2x)$$

$$\frac{dy}{y^2} = \frac{\cancel{y^2}(6-2x)\,dx}{\cancel{y^2}}$$

$$\int y^{-2}\,dy = \int 6-2x\,dx$$

$$-1y^{-1} = 6x - x^2 + C$$

$(3, 1/4)$ — x, y

$$\frac{-1}{1/4} = 6(3) - (3)^2 + C$$

$$-4 = 18 - 9 + C$$

$$-4 = 9 + C$$

$$C = -13$$

$$\frac{-1}{y} = 6x - x^2 - 13$$

$$\frac{1}{y} = -6x + x^2 + 13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

$$\boxed{f(x) = \frac{1}{x^2 - 6x + 13}}$$

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) Tangent to graph means $f'(x) = 0$

$\frac{dy}{dx} = \frac{3-X}{Y} \rightarrow \frac{3-X}{-2} = 0$

deriv.

$3 - X = 0$

$X = 3$

$\boxed{\text{Point} \rightarrow (3,-2)}$

Use 2nd Deriv. Test to determine if local max or min $\quad \frac{dy}{dx} = (3-x)y^{-1}$

$\frac{d^2y}{dx^2} = (-1)(y^{-1}) - (3-x)(y^{-2}\frac{dy}{dx})$

$= \frac{-1}{Y} - \frac{3-X}{Y^2}(3-x)(y^{-1})$

Plug in $(3,-2)$

$= -\frac{1}{-2} - \cancel{\frac{3-3}{(-2)^2}(3-3)}(-2^{-1})$

$= \frac{1}{2}$

Since $\frac{d^2y}{dx^2}$ at $(3,-2)$ is >0 then $y = f(x)$ is concave up which means $(3,-2)$ is relative min.

(b) $\frac{dY}{dX} = \frac{3-X}{Y}$

$\int Y\,dy = \int 3 - x\,dX$

$\frac{1}{2}y^2 = 3X - \frac{1}{2}X^2 + C$

$(6,-4)$
$\;\,x\;\;\;y$

$\frac{1}{2}(-4)^2 = 3(6) - \frac{1}{2}(6)^2 + C$

$8 = 18 - 18 + C$

$C = 8$

$\frac{1}{2}y^2 = 3X - \frac{1}{2}X^2 + 8$

$y^2 = 6X - X^2 + 16$

$y = \pm\sqrt{6X - X^2 + 16}$

$(6,-4)$

neg

$\boxed{f(x) = -\sqrt{6X - X^2 + 16}}$