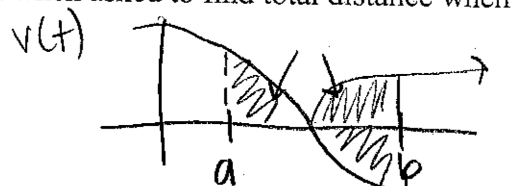
AP Calculus
Unit 6 – Basic Integration & Applications

Day 5 Notes: The Fundamental Theorem of Calculus, Particle Motion, and Average Value

(1) $\int_a^b v(t)dt = p(b) - p(a)$ where $v(t)$ represents the velocity and $p(t)$ represents the position.

(2) $\int_a^b v(t)dt =$ The Net Distance the particle travels on the interval from $t = a$ to $t = b$. If $v(t) > 0$ on the interval (a, b), then it also represents the Total Distance.

(3) $\int_a^b |v(t)|dt =$ The Total Distance the particle travels on the interval (a, b), whether or not $v(t) > 0$. To be safe, always do this integral when asked to find total distance when given velocity.



Example 1: The velocity of a particle that is moving along the x – axis is given by the function $v(t) = 3t^2 + 6$. (This is a non-calculator active question.)

a. If the position of the particle at $t = 4$ is 72, what is the position when $t = 2$?

$$\int_2^4 v(t)\,dt = p(4) - p(2)$$
$$\int_2^4 3t^2 + 6dt = 72 - p(2)$$
$$t^3 + 6t + C\Big]_2^4 = 72 - p(2)$$
$$[(4)^3 + 6(4)] - [(2)^3 + 6(2)] = 72 - p(2)$$
$$88 - 20 = 72 - p(2)$$

$$68 = 72 - p(2)$$
$$-4 = -p(2)$$
$$\boxed{p(2) = 4}$$

b. What is the total distance the particle travels on the interval $t = 0$ to $t = 7$?

$$\text{Total Distance} = \int_0^7 3t^2 + 6$$
$$t^3 + 6t + C\Big]_0^7$$
$$[(7)^3 + 6(7)] - [(0)^3 + 6(0)]$$
$$343 + 42 - 0$$
$$= \boxed{385}$$

Example 2: The velocity of a particle that is moving along the x – axis is given by the function $v(t) = 0.5e^t(t-2)^3$. (This is a calculator active question.)

a. If the position of the particle at $t = 1.5$ is 2.551, what is the position when $t = 3.5$?

math 9

$$\int_{1.5}^{3.5} 0.5e^t(t-2)^3 = p(3.5) - p(1.5)$$

$$15.919 = p(3.5) - 2.551$$

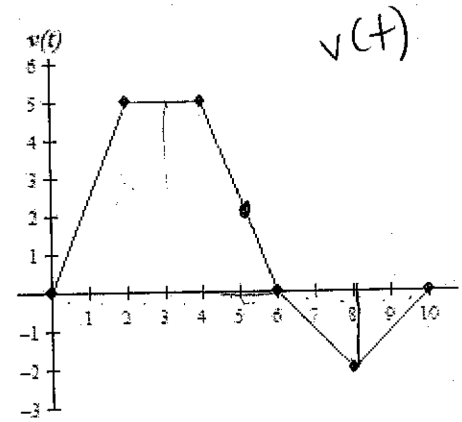
$$\boxed{18.47 = p(3.5)}$$

b. What is the total distance that the object travels on the interval $t = 1$ to $t = 5$?

math 9

$$\int_1^5 |0.5e^t(t-2)^3| = \boxed{913.067}$$

Example 3: The graph of the velocity, measured in feet per second, of a particle moving along the x – axis is pictured below. The position, $p(t)$, of the particle at $t = 8$ is 12. Use the graph of $v(t)$ to answer the questions that follow.



a. What is the position of the particle at $t = 3$?

$$\int_3^8 v(t)\,dt = p(8) - p(3)$$

$$\downarrow$$

$$(1)(5) + \tfrac{1}{2}(2)(5) + \tfrac{1}{2}(2)(-2) =$$

$$5 + 5 - 2 =$$

$$8 = 12 - p(3)$$

$$-4 = p(3) \qquad \boxed{p(3) = 4 \text{ ft}}$$

b. What is the acceleration when $t = 5$?

$$a(5) = v'(5) = \frac{v(4) - v(6)}{4 - 6}$$

$$= \frac{5 - 0}{-2} = \boxed{-2.5 \text{ ft/sec}^2}$$

c. What is the net distance the particle travels from $t = 0$ to $t = 10$?

$$\int_0^{10} v(t)\,dt =$$

$$\tfrac{1}{2}(5)(2+6) - \tfrac{1}{2}(4)(2)$$

$$= 20 - -4 = \boxed{16 \text{ ft}}$$

d. What is the total distance the particle travels from $t = 0$ to $t = 10$?

$$\int_0^{10} |v(t)|\,dt = \tfrac{1}{2}(5)(2+6) + \tfrac{1}{2}(4)(2)$$

$$= 20 + 4 = \boxed{24 \text{ ft}}$$

t	0	3	6	9	12	15	18
$V(t)$	2.3	2.7	2.0	1.3	1.0	1.7	2.1

Example 4: The table above shows values of the velocity, $V(t)$ in meters per second, of a particle moving along the x – axis at selected values of time, t seconds.

a. What does the value of $\int_0^{18} V(t)dt$ represent? net distance the particle travels from $t=0$ to $t=18$. Since all values of $v(t)>0$, it also represents the total distance.

b. Using a left Riemann sum of 6 subintervals of equal length, estimate the value of $\int_0^{18} V(t)dt$. Indicate units of measure.

$$3[2.3+2.7+2.0+1.3+1.0+1.7]$$
$$\approx \boxed{33 \text{ meters}}$$

c. Using a right Riemann sum of 6 subintervals of equal length, estimate the value of $\int_0^{18} V(t)dt$. Indicate units of measure.

$$3[2.7+2.0+1.3+1.0+1.7+2.1] = \boxed{32.4 \text{ meters}}$$

d. Using a midpoint Riemann sum of 3 subintervals of equal length, estimate the value of $\int_0^{18} V(t)dt$. Indicate units of measure.

$$6[2.7+1.3+1.7] = \boxed{34.2 \text{ meters}}$$

e. Using a trapezoidal sum of 6 subintervals of equal length, estimate the value of $\int_0^{18} V(t)dt$. Indicate units of measure.

$$\boxed{32.7 \text{ meters}}$$

$$\tfrac{1}{2}(3)(2.3+2.7)+\tfrac{1}{2}(3)(2.7+2.0)+\tfrac{1}{2}(3)(2.0+1.3)+\tfrac{1}{2}(3)(1.3+1)+\tfrac{1}{2}(3)(1+1.7)+\tfrac{1}{2}(3)(1.7+2.1)$$
$$7.5+7.05+4.95+3.45+4.05+5.7$$

f. Find the average acceleration of the particle from $t = 3$ to $t = 9$. For what value of t, in the table, is this average acceleration approximately equal to $v'(t)$? Explain your reasoning

$$\text{avg accer} = \frac{v(3)-v(9)}{3-9} = \frac{2.7-1.3}{-6} = \boxed{-0.233 \text{ m/s}^2}$$

According to the mean value Theorem, the avg. acceleration should be approximately equal to $v'(6)$.

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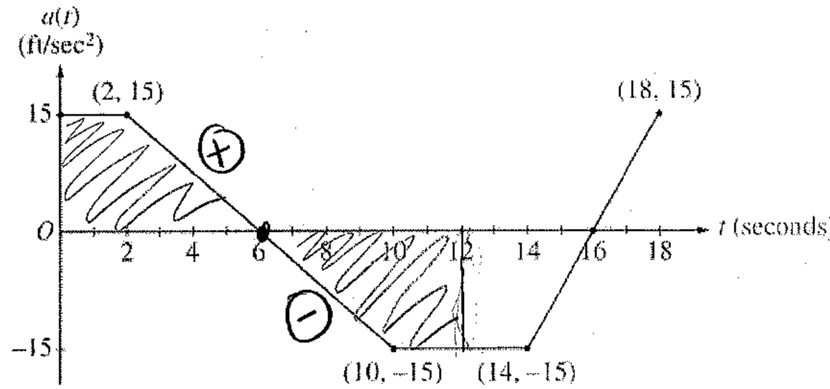
Problem #3



A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.

(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $a(2) > 0$, then $v(t)$ is increasing at $t = 2$.

(b) $\int_0^k a(t) = v(k) - v(0) = 0$
$v(k) = v(0)$ $\rightarrow$ $\int_0^{12} a(t)\,dt = \frac{1}{2}(15)(2+6) - \frac{1}{2}(15)(2+6) = 0$

$\boxed{t=12}$

(c) $t=6$ absolute maximum velocity

$v(6) = 55 + \int_0^6 a(t)\,dt = 55 + \frac{1}{2}(15)(2+6) = \boxed{115 \text{ ft/sec}}$

$\int_6^{18} a(t)\,dt < 0$ so $v(18) < v(6)$

(d) $v(16)$ minimum

$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - \frac{1}{2}(15)(10+4) = 10 > 0$

$\therefore$ velocity is never equal to 0.