Day 2 Notes: Riemann Sums

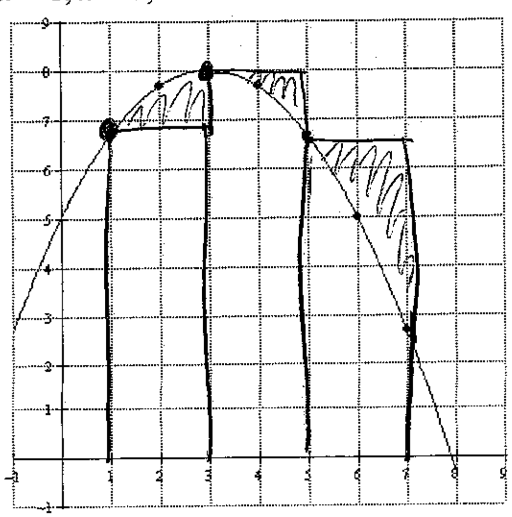
In calculus, the result of $\int f(x)dx$ is a function that represents the anti-derivative of the function $f(x)$. This is also sometimes referred to as an INDEFINITE INTEGRAL.

The result of $\boxed{\int_a^b f(x)dx}$ is a value that represents the area of the region bounded by the curve of $f(x)$ and the x – axis on the interval $a \le x \le b$.

Calculating Riemann sums is a way to **estimate** the area under a curve, the value of $\int_a^b f(x)dx$, for a graphed function on a particular interval. In this activity, you will learn to calculate four types of Riemann sums: Left Hand, Right Hand, Midpoint, and Trapezoidal Sums.

Approximation #1 – Left Hand Riemann Sum with intervals of length 2 units

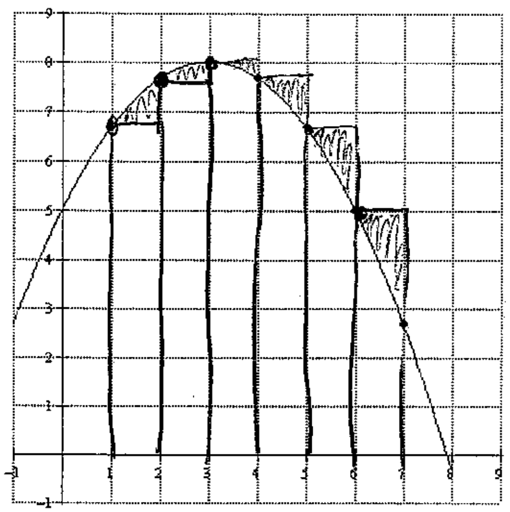
Let's consider for a moment the function $f(x) = -\frac{1}{3}x^2 + 2x + 5$. This function is graphed below. On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length 2 units. Place the upper left hand vertex of the rectangle on the curve each time. Then, calculate the area of each rectangle and sum the areas to *approximate* the area of the region under the curve bounded by $f(x) = -\frac{1}{3}x^2 + 2x + 5$, $x = 1$, $x = 7$, and the x-axis.



$$\int_1^7 f(x)dx \approx 2 \cdot f(1) + 2 \cdot f(3) + 2f(5)$$
$$\approx 2\left[f(1) + f(3) + f(5)\right]$$
$$\approx 2\left[\frac{20}{3} + 8 + \frac{20}{3}\right]$$
$$\approx \boxed{42.667}$$

Approximation #2 – Left Hand Riemann Sum with intervals of length ***1 unit***

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length1 unit.



$$\int_1^7 f(x)\,dx \approx$$

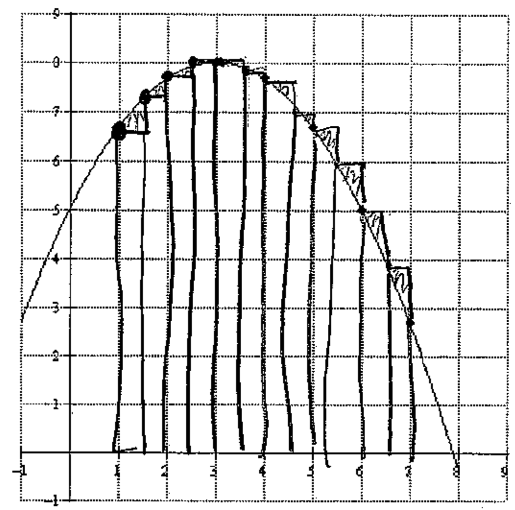
$$\approx 1\left[f(1)+f(2)+f(3)+f(4)+f(5)+f(6)\right]$$

$$\approx 1\left[\frac{20}{3}+\frac{23}{3}+8+\frac{23}{3}+\frac{20}{3}+5\right]$$

$$\approx 41.667$$

Approximation #3 – Left Hand Riemann Sum with intervals of length ***½ unit***

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length ½ unit.



$$\int_1^7 f(x)\,dx \approx$$

$$\approx \frac{1}{2}\left[f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)+f(4)+f(4.5)+f(5)+f(5.5)+f(6)+f(6.5)\right]$$

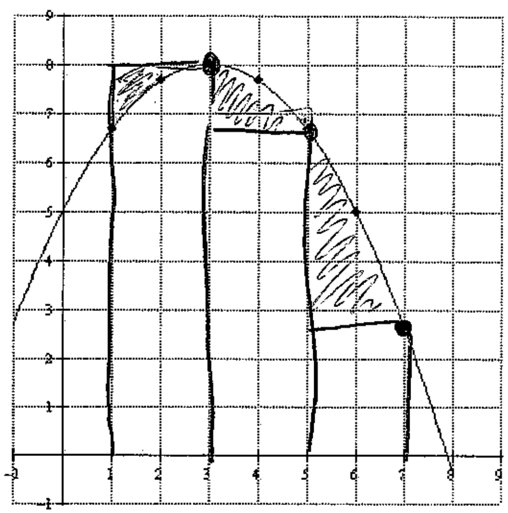
$$\approx \frac{1}{2}\left[\frac{20}{3}+7.25+\frac{23}{3}+7.917+8+7.917+\frac{23}{3}+7.25+\frac{20}{3}+5.917+5+3.917\right]$$

$$\approx 40.917$$

Now, we are going to change things up a little bit. On the next three approximations, we are going to do a right hand sum. That is, draw your rectangles so that the upper right hand vertex of the rectangle is on the curve of the function.

Approximation #4 – Right Hand Riemann Sum with intervals of length 2 units

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length 2 units.



$$\int_1^7 f(x)\,dx \approx$$

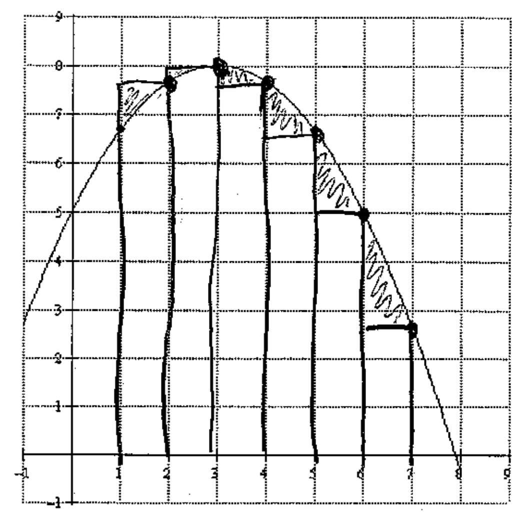
$$2[f(3)+f(5)+f(7)]$$

$$\approx 2\left[8+\frac{20}{3}+\frac{8}{3}\right]$$

$$\approx 34.667$$

Approximation #5 – Right Hand Riemann Sum with intervals of length 1 unit

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length 1 unit.



$$\int_1^7 f(x)\,dx \approx$$

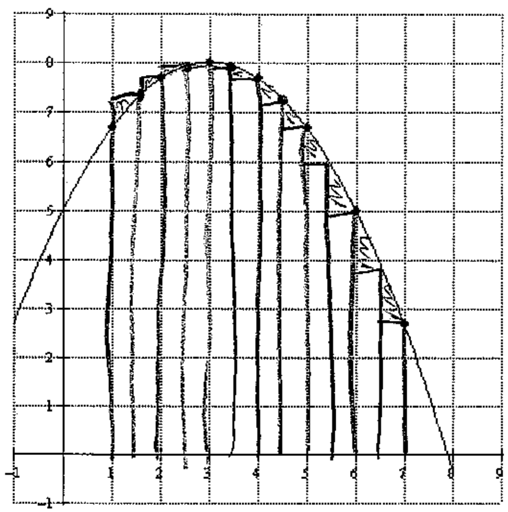
$$1[f(2)+f(3)+f(4)+f(5)+f(6)+f(7)]$$

$$\approx \left[\frac{23}{3}+8+\frac{23}{3}+\frac{20}{3}+5+\frac{8}{3}\right]$$

$$\approx 37.667$$

Approximation #6 – Right Hand Riemann Sum with intervals of length ½ unit

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length ½ unit .



$$\int_1^7 f(x)\,dx =$$

$$\frac{1}{2}\left[f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)+f(4)+f(4.5)+f(5)+f(5.5)+f(6)+f(6.5)+f(7)\right]$$

$$\approx \frac{1}{2}\left[7.25+\frac{23}{3}+7.917+8+7.917+\frac{23}{3}+7.25+\frac{20}{3}+5.917+5+3.917+\frac{8}{3}\right]$$

$$\approx 38.917$$

Before we continue, what inference can you make about the approximations as the lengths of the rectangles decreases?

As the length of the rectangles decrease, the closer the approximation of the area gets to the actual area.

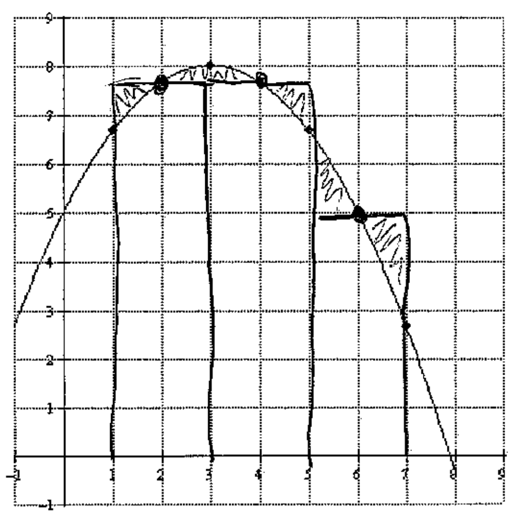
Graphically, why do you suppose this is so?

The smaller the length of the rectangle, the less "overage/underage" area there seems to be.

On the next two approximations, we are going to do a midpoint sum. That is, draw your rectangles so that the midpoint of the upper left and right hand vertices of the rectangle is on the curve of the function.

Approximation #7 – Midpoint Riemann Sum with intervals of length 2 units

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length 2 units.



$\int_1^7 f(x)\,dx$

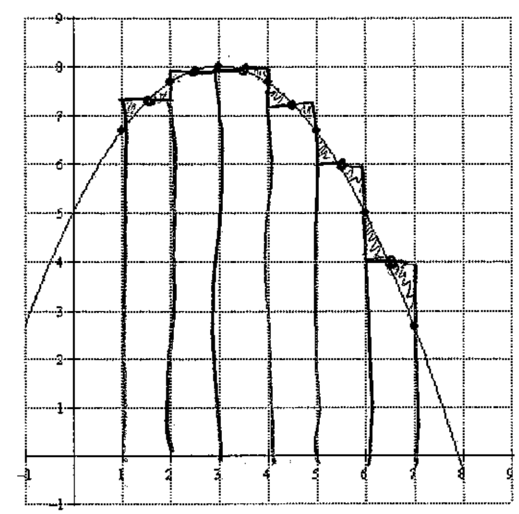
$\approx 2\left[f(2) + f(4) + f(6)\right]$

$\approx 2\left[\frac{23}{3} + \frac{23}{3} + 5\right]$

≈ 40.667

Approximation #8 – Midpoint Riemann Sum with intervals of length 1 units

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into rectangles of length 1 unit.



$\int_1^7 f(x)\,dx$

$\approx 1\left[f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5) + f(6.5)\right]$

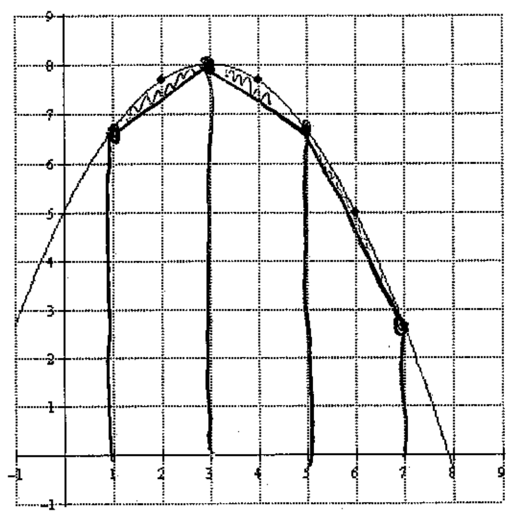
$\approx 1\left[7.25 + 7.917 + 7.917 + 7.25 + 5.917 + 3.917\right]$

≈ 40.168

On the next two approximations, we are going to draw trapezoids between the curve and the x – axis to approximate the area. Remember that the area of a trapezoid is found using the formula $A = \frac{1}{2}h(b_1 + b_2)$, where b_1 and b_2 are the parallel sides, and h is the distance between those parallel sides.

Approximation #9 – Trapezoidal Riemann Sum with intervals of length 2 units

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into trapezoids of height 2 units.



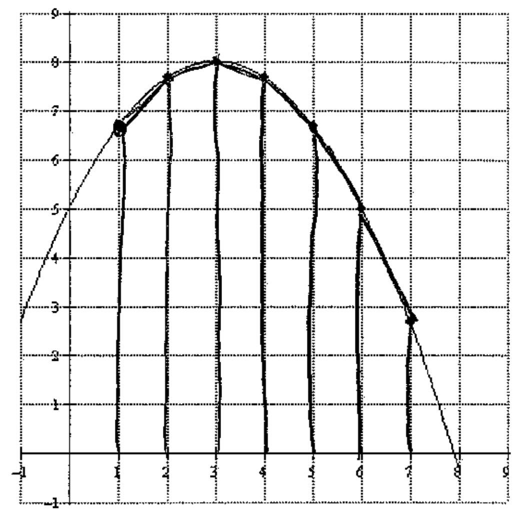
$$\int_1^7 f(x)\,dx \approx \cancel{\tfrac{1}{2}}\cancel{(2)}[f(1)+f(3)] + \cancel{\tfrac{1}{2}}\cancel{(2)}[f(3)+f(5)] + \cancel{\tfrac{1}{2}}\cancel{(2)}[f(5)+f(7)]$$

$$\approx \left[\frac{20}{3}+8\right] + \left[8+\frac{20}{3}\right] + \left[\frac{20}{3}+\frac{8}{3}\right]$$

$$\approx 38.667$$

Approximation #10 – Trapezoidal Riemann Sum with intervals of length 1 unit

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into trapezoids of height 1 unit.



$$\int_1^7 f(x)\,dx = \tfrac{1}{2}(1)[f(1)+f(2)] + \tfrac{1}{2}(1)[f(2)+f(3)] + \tfrac{1}{2}(1)[f(3)+f(4)] + \tfrac{1}{2}(1)[f(4)+f(5)]$$

$$+ \tfrac{1}{2}(1)[f(5)+f(6)] + \tfrac{1}{2}(1)[f(6)+f(7)]$$

$$\approx \tfrac{1}{2}[f(1)+2f(2)+2f(3)+2f(4)+2f(5)+2f(6)+f(7)]$$

$$\approx \tfrac{1}{2}\left[\tfrac{20}{3} + 2\left(\tfrac{23}{3}\right) + 2(8) + 2\left(\tfrac{23}{3}\right) + 2\left(\tfrac{20}{3}\right) + 2(5) + \tfrac{8}{3}\right]$$

$$\approx 39.667$$

Type of Sum	**Numerical Approximation**	**Do you think this is an over or under approximation of the area?**
Left Hand Riemann Sum Intervals of 2 units	42.667	over
Left Hand Riemann Sum Intervals of 1 unit	41.667	over
Left Hand Riemann Sum Intervals of ½ unit	40.917	over
Right Hand Riemann Sum Intervals of 2 units	34.667	under
Right Hand Riemann Sum Intervals of 1 unit	37.667	under
Right Hand Riemann Sum Intervals of ½ unit	38.917	under
Midpoint Riemann Sum Intervals of 2 units	40.667	over
Midpoint Riemann Sum Intervals of 1 unit	40.168	over
Trapezoidal Riemann Sum Intervals of 2 units	38.667	under
Trapezoidal Riemann Sum Intervals of 1 unit	39.667	under

At this point, place a star next to the approximation that you feel is the most accurate to the actual area.

In our next lesson, we will learn how to find the EXACT area of a region between the graph of a function and the x – axis.

In the space below, we will come back to this to find the exact area once we complete the next lesson in order to see which approximation to the left is the most accurate.

Given the table of values below, approximate each definite integral by finding the indicated Riemann Sum.

a. Approximate $\int_0^{25} f(x)dx$ using a midpoint sum and three subintervals.

x	0	4	7	12	15	20	25
$f(x)$	15	6	–5	–10	–2	8	20

$$\approx 7(6) + 8(-10) + 10(8) \approx \boxed{42}$$

b. Approximate $\int_0^{15} f(x)dx$ using a left hand sum and four subintervals.

x	0	4	7	12	15	~~20~~	~~25~~
$f(x)$	15	6	–5	–10	–2	~~8~~	~~20~~

$$\approx 4(15) + 3(6) + 5(-5) + 3(-10) \approx \boxed{23}$$

c. Approximate $\int_4^{20} f(x)dx$ using a right hand sum and four subintervals.

x	~~0~~	4	7	12	15	20	~~25~~
$f(x)$	~~15~~	6	–5	–10	–2	8	~~20~~

$$\approx 3(-5) + 5(-10) + 3(-2) + 5(8) \approx \boxed{-31}$$

d. Approximate $\int_0^{25} f(x)dx$ using a trapezoidal sum and three subintervals

x	0	4	7	12	15	20	25
$f(x)$	15	6	–5	–10	–2	8	20

$$\approx \frac{1}{2}(7)[15+-5] + \frac{1}{2}(8)[-5+-2] + \frac{1}{2}(10)[-2+20]$$

$$35 + -28 + 90$$

$$\approx \boxed{97}$$