AP Calculus
Unit 5 – Applications of the Derivative – Part 2

Day 4 & 5 Notes: Particle Motion Problems

Average and Instantaneous Velocity

Suppose $p(t)$ is a function which defines the position of a moving object at time, t.

Average Velocity = Avg. Rate of change of Position = $\frac{p(a)-p(b)}{a-b}$

Instantaneous Velocity = $p'(a)$

Example 1: A particle's position is given by the function $p(t) = e^t \sin t$, where $p(t)$ is measured in centimeters and t is measured in seconds. Answer the following questions.

a) What is the average velocity on the interval $t = 1$ to $t = 3$ seconds? Indicate appropriate units of measure.

Avg Velocity $= \frac{p(1)-p(3)}{1-3} = \frac{2.287-2.834}{-2} = \boxed{0.274 \text{ cm/sec}}$

b) What is the instantaneous velocity of the particle at time $t = 1.5$. Indicate appropriate units of measure. $p'(a)$

$p(t) = e^t \sin t$

$p'(t) = (e^t)(\sin t) + (e^t)(\cos t)$ $\rightarrow$ $p'(1.5) = (e^{1.5})(\sin 1.5) + (e^{1.5})(\cos 1.5)$

$= \boxed{4.787 \text{ cm/sec}}$

Average and Instantaneous Acceleration

If $p(t)$ is position & $p'(t) = v(t)$, the velocity, then...

Average Acceleration = $\frac{v(a)-v(b)}{a-b}$

Instantaneous Acceleration = $v'(a) = p''(a)$

c) What is the average acceleration on the interval $t = 1$ to $t = 3$ seconds? Indicate appropriate units of measure.

$p'(t) = v(t) = e^t \sin t + e^t \cos t$

$\frac{v(1)-v(3)}{1-3} = \frac{3.756-(-17.050)}{-2}$

$= \frac{\text{cm/sec}}{\text{sec}}$ $= \boxed{-10.403 \text{ cm/sec}^2}$

d) What is the instantaneous acceleration of the particle at time $t = 1.5$.

$p''(t) = v'(t) = \cancel{e^t(\sin t)} + (e^t)(\cos t) + (e^t)(\cos t) + \cancel{(e^t)(-\sin t)}$

$= 2e^t \cos t$

$v'(1.5) = 2e^{1.5}(\cos 1.5) = \boxed{0.634 \text{ cm/sec}^2}$

In summary, let's correlate the concepts of position, velocity, and acceleration to what we already know about a function and its first and second derivative.

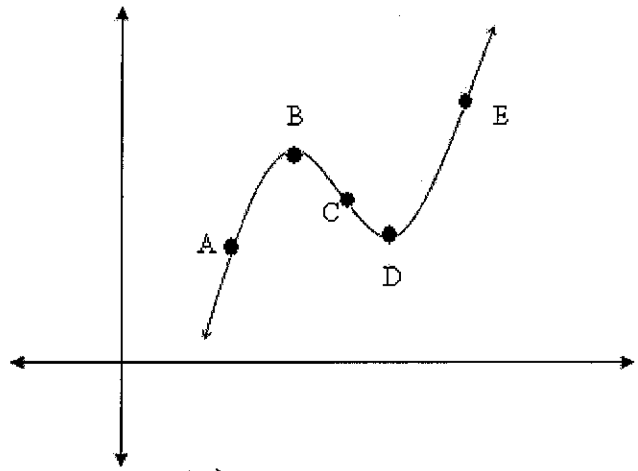
$f(x)$	corresponds with	$p(t)$ = position
$f'(x)$	corresponds with	$p'(t) = v(t)$ = velocity
$f''(x)$	corresponds with	$p''(t) = v'(t) = a(t)$ = acceleration

Let's summarize our relationships between position, velocity and acceleration below.

Velocity $p'(t)$	***Position (Motion of the Particle)*** $p(t)$
Is = 0 or is undefined	potential change in direction of moving object
Is > 0	position is increasing → particle moving to Right
Is < 0	position is decreasing → particle moving to left
Changes from positive to negative	particle changes from moving Right to moving left.
Changes from negative to positive	particle changes from moving left to moving right.

Acceleration $p''(t)$	***Velocity*** $p'(t)$
Is = 0 or is undefined	potential change in sign of velocity
Is > 0	is increasing
Is < 0	is decreasing
Changes from positive to negative	changes from increasing to decreasing
Changes from negative to positive	changes from decreasing to increasing

The graph below represents the position, $s(t)$, of a particle which is moving along the x axis.



$s(t)$
position

$p'(t)=0$

- At which point(s) is the velocity equal to zero? Justify your answer.

$v(t)=0$ at $\boxed{t=B}$ and $\boxed{t=D}$ b/c $s(t)$ changes directions.

- At which point(s) does the acceleration equal zero? Justify your answer.

$p''(t)=0$

$a(t)=0$ at $\boxed{t=C}$ b/c $s(t)$ has a point of inflection

- On what interval(s) is the particle's velocity positive? Justify your answer.

$p'(t)>0$

$v(t)>0$ on $\boxed{(0,B)\cup(D,\infty)}$ b/c $s(t)$ is increasing.

- On what interval(s) is the particle's velocity negative? Justify your answer.

$p'(t)<0$

$v(t)<0$ on $\boxed{(B,D)}$ b/c $s(t)$ is decreasing.

- On what interval(s) is the particle's acceleration positive? Justify your answer.

$p''(t)>0$

$a(t)>0$ on (C,∞) b/c $s(t)$ is concave up.

- On what interval(s) is the particle's acceleration negative? Justify your answer.

$p''(t)<0$

$a(t)<0$ on $(0,C)$ b/c $s(t)$ is concave down.

Five Commandments of Particle Motion → Horizontal Movement

1. If the velocity is positive, then object is moving to the Right.

2. If the velocity is negative, then object is moving to the left.

3. The speed is increasing when the signs of velocity & acceleration are the same.

4. The speed is decreasing when the signs of velocity & acceleration are opposite.

5. If the velocity is equal to 0 but the acceleration is not $= 0$, then the object is momentarily stopped & changing directions.

Example 2: Suppose the velocity of a particle is given by the function $v(t) = (t+2)(t+4)^2$ for $t \geq 0$, where t is measured in minutes and $v(t)$ is measured in inches per minute. Answer the questions that follow.

a) Find the values of $v(3)$ and $v'(3)$. Based on these values, describe the speed of the particle at $t = 3$.

$v(3) = (3+2)(3+4)^2 = (5)(49) = \boxed{245 \text{ in/min}}$

$v'(t) = (1)(t+4)^2 + (t+2)[2(t+4)'(1)]$

$= (t+4)^2 + 2(t+2)(t+4)$

$v'(3) = (3+4)^2 + 2(3+2)(3+4) = 49 + 70 = \boxed{119 \text{ in/min}^2}$

Since $v(3) > 0$ and $v'(3) > 0$, then the speed is increasing at $t = 3$.

b) On what interval(s) is the particle moving to the left? Right? Show your analysis and justify your answer.

$v(t) = (t+2)(t+4)^2 = 0$

$t = -2 \quad t = -4$

[t can't be negative]

$t = 1$

0 ——— (+)(+) pos. ——→

For $t \geq 0$, $v(t)$ is always > 0, so the particle is always moving to the Right.