AP Calculus
Unit 5 – Applications of the Derivative – Part 2

Day 9 Notes: L'Hôpital's Rule

In the same sense, L'Hôpital's Rule uses derivatives to find limits of certain functions, in particular, limits that are of indeterminant form. In our study of limits, we encountered limits of functions that were of this form. When direct substitution is applied to evaluate a limit, results of $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{-\infty}{-\infty}$, or $\frac{\infty}{-\infty}$ are referred to as indeterminant form.

Let's take a look at an example of those from earlier in the year and remember the methods we used to find the values of these limits.

Try to evaluate the limit below by direct substitution of $x = 3$.	Direct substitution gave us the indeterminant form of $\frac{0}{0}$. Algebraically, factor and cancel and then find the limit.
$\lim_{x\to 3} \frac{2x^2 - x - 15}{x^2 - x - 6}$ $\frac{2(3)^2 - 3 - 15}{(3)^2 - 3 - 6} = \frac{0}{0}$	$\lim_{x\to 3} \frac{(2x+5)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \frac{2(3)+5}{3+2}$ $= \boxed{\frac{11}{5}}$

L'Hôpital's Rule:

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ OR $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Use L'Hôpital's Rule to find $\lim_{x\to 3} \frac{2x^2 - x - 15}{x^2 - x - 6}$ in the space below. Compare your result to the result above.

$$\lim_{x\to 3} \frac{4x-1}{2x-1} = \frac{4(3)-1}{2(3)-1} = \boxed{\frac{11}{5}}$$

Two other limits that we learned earlier were also of indeterminant form. Numericallly, we investigated the limits below to find their values.

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \text{ and } \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

In the boxes below, use L'Hôpital's Rule to validate the numerical derivation of the values of the limits above.

$\lim_{x\to 0}\frac{\sin x}{x} = \frac{0}{0}$ $\lim_{x\to 0}\frac{\cos x}{1} = \cos(0) = \boxed{1}$	$\lim_{x\to 0}\frac{1-\cos x}{x} = \frac{0}{0}$ $\lim_{x\to 0}\frac{\sin x}{1} = \sin(0) = \boxed{0}$

Use L'Hôpital's Rule to find the values of each of the following limits.

$\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{x-\frac{\pi}{2}} = \frac{0}{0}$ $\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{1}$ $=\sin\left(\frac{\pi}{2}\right)$ $=\boxed{1}$	$\lim_{x\to 0}\frac{7x-\sin x}{x^2+\sin(3x)} = \frac{0}{0}$ $\lim_{x\to 0}\frac{7-\cos x}{2x+3\cos(3x)}$ $=\frac{7-\cos(0)}{2(0)+3\cos(0)}$ $=\frac{6}{3}=\boxed{2}$	$\lim_{x\to 0}\frac{\sin x\cos x}{x+\sin(2x)} = \frac{0}{0}$ $\lim_{x\to 0}\frac{(\cos x)(\cos x)+(\sin x)(-\sin x)}{1+2\cos(2x)}$ $=\frac{(\cos 0)^2-(\sin 0)^2}{1+2\cos(0)}$ $=\boxed{\frac{1}{3}}$

This example will require multiple applications of L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{2\cos x-2\cos(2x)}{1-\cos x} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{-2\sin x+4\sin(2x)}{\sin x} = \frac{-2\sin(0)+4\sin(0)}{\sin(0)} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{-2\cos x+8\cos(2x)}{\cos x} = \frac{-2\cos(0)+8\cos(0)}{\cos(0)} = \frac{-2+8}{1} = \boxed{6}$$

Now, let's remember how we evaluated limits at infinity. Consider the limit below.

Algebraically find the limit below as we did earlier in the course.	Using your knowledge of the existence of horizontal asymptotes, validate your answer for the limit to the left.
$$\lim_{x\to\infty} \frac{4x^2-5x}{1-3x^2}$$ $$\frac{\frac{4x^2}{x^2}-\frac{5x}{x^2}}{\frac{1}{x^2}-\frac{3x^2}{x^2}}$$ $$\lim_{x\to\infty} \frac{4-\frac{5}{x}}{\frac{1}{x^2}-3} = \frac{4-0}{1-3} = \boxed{-\frac{4}{3}}$$	degrees are the same → $\boxed{-\frac{4}{3}}$

Now, use L'Hôpital's Rule to find $\lim_{x\to\infty} \frac{4x^2-5x}{1-3x^2}$.

$$\lim_{x\to\infty} \frac{8x-5}{-6x} = \frac{\infty-5}{-\infty} = \frac{\infty}{-\infty}$$

$$\lim_{x\to\infty} \frac{8}{-6} = \boxed{-\frac{4}{3}}$$

Use L'Hôpital's Rule to find $\lim_{x\to\infty} \frac{5x^2-x}{3x^3-2x^2-3x}$ and then, using your knowledge of horizontal asymptotes of rational functions from pre-calculus, explain why the result is what it is.

$$\lim_{x\to\infty} \frac{10x-1}{9x^2-4x-3} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{10}{18x-4} = \frac{10}{\infty} = \boxed{0}$$

$\frac{\text{Small degree}}{\text{Large degree}}$ means horiz. asympt is 0.

Use L'Hôpital's Rule to find each of the following limits.

$\frac{\infty}{\infty} =$ $\lim_{x\to\infty} \frac{\ln(2x)}{2\sqrt{x}}$ $= \lim_{x\to\infty} \frac{\frac{2}{2x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^{1/2}}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{x^{1/2}}{1}$

$1/2x^{1/2}$

$= \lim_{x\to\infty} \frac{1}{x^{1/2}} = \frac{1}{\infty} = \boxed{0}$

$\frac{0}{0} =$ $\lim_{x\to\frac{1}{2}} \frac{x\cos(\pi x)}{e^x - \sqrt{e}}$ $= \lim_{x\to 1/2} \frac{(1)(\cos(\pi x)) + (x)(-\sin(\pi x)(\pi)}{e^x}$

$= \frac{\cos(\frac{\pi}{2}) + (\frac{1}{2})(-\sin(\frac{\pi}{2})(\pi)}{e^{1/2}}$

$= \frac{0 + (\frac{1}{2})(-1)(\pi)}{e^{1/2}} = \frac{-\frac{\pi}{2}}{e^{1/2}} = \frac{-\pi}{2} \cdot \frac{1}{e^{1/2}} = \boxed{\frac{-\pi}{2\sqrt{e}}}$

$\frac{0}{0} =$ $\lim_{x\to 0} \frac{1-e^x}{\ln(2-e^x)}$ $= \lim_{x\to 0} \frac{-e^x}{\frac{-e^x}{2-e^x}}$

$= \lim_{x\to 0} \cancel{-e^x} \cdot \frac{2-e^x}{\cancel{-e^x}} = \lim_{x\to 0} 2-e^x$

$= 2 - e^0$

$= 2 - 1$

$= \boxed{1}$

$\frac{0}{0} =$ $\lim_{x\to 0} \frac{x\cos x}{(x+1)\sin x}$ $= \lim_{x\to 0} \frac{(1)(\cos x) + (x)(-\sin x)}{(1)(\sin x) + (x+1)(\cos x)}$

$= \frac{\cos(0) + (0)(-\sin(0))}{(\sin 0) + (0+1)(\cos(0))}$

$= \frac{1+0}{0+1} = \frac{1}{1} = \boxed{1}$