AP Calculus
Unit 5 – Applications of the Derivative – Part 2

Day 3 Notes: Applying Theorems in Calculus (Intermediate Value, Extreme Value, Mean Value, Rolle's)

Example 1: The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table below shows the rate as measured every 3 hours for a 24-hour period.

t (hours)	0	3	6	9	12	15	18	21	24
$R(t)$ (gallons per hour)	9.6	10.4	10.8	11.2	11.4	11.3	10.7	10.2	9.6

a. Estimate the value of $R'(5)$, indicating correct units of measure. Explain what this value means about $R(t)$.

$$R'(5) \approx \frac{R(3)-R(6)}{3-6} = \frac{10.4-10.8}{3-6} = \boxed{0.133}$$

Since $R'(5) > 0$, then the rate at which water is flowing out of the pipe is increasing at $t = 5$ hrs.

b. Using correct units of measure, find the average rate of change of $R(t)$ from $t = 3$ to $t = 18$.

$$\frac{R(3)-R(18)}{3-18} = \frac{10.4-10.7}{3-18} = \boxed{0.02 \text{ gallons per hour}}$$

Rolle's

c. Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

→ $R(t)$ is continuous on $[0, 24]$ ✓

→ $R(t)$ is differentiable on $(0, 24)$ ✓

→ Since $R(0) = 9.6$ & $R(24) = 9.6$, then Rolle's Theorem guarantees a value of t on $0 < t < 24$ such that $R'(t) = 0$.

Calc. Active

Example 2: The total order and transportation cost $C(x)$, measured in dollars, of bottles of Pepsi Cola is approximated by the function

$$C(x) = 10{,}000\left(\frac{1}{x} + \frac{x}{x+3}\right),$$

where x is the order size in number of bottles of Pepsi Cola in hundreds. Answer the following questions.

a. Is there guaranteed a value of r on the interval $0 \le r \le 3$ such that the average rate of change of cost is equal to $C'(r)$? Give a reason for your answer. M.V.T

$x \neq 0$
$x \neq -3$

$C(0)$ is undefined $\rightarrow$ Therefore, $C(x)$ is not continuous on $[0,3]$.

$\therefore$ Mean Value Theorem does not guarantee a value of r on $(0,3)$ such that $C'(r) = \frac{C(0)-C(3)}{0-3}$.

b. Is there a value of r on the interval $3 \le r \le 6$ such that $C'(r) = 0$? Give a reason for your answer and if such a value of r exists, then find that value of r. Rolle's

$\rightarrow$ $C(x)$ is only discontinuous at $x=0$ and $x=-3$

$\therefore$ $C(x)$ is continuous on $[3,6]$. ✓

$\rightarrow$ $C(x)$ is differentiable on $(3,6)$ ✓

$\rightarrow$ $C(3) = 10{,}000\left(\frac{1}{3} + \frac{3}{3+3}\right) = 8333.333$

$C(6) = 10{,}000\left(\frac{1}{6} + \frac{6}{6+3}\right) = 8333.333$ $\rangle$ $C(3) = C(6)$ ✓

$\therefore$ Rolle's Theorem is applicable

Rolle's Theorem guarantees a value of r such that $C'(r) = 0$.

$\boxed{r = 4.098}$

Xmin = 3
Xmax = 6
Ymin = 8000
Ymax = 8500

*Graph $C(x)$ $\rightarrow$ Find relative max or rel. min on $[3,6]$

c. For $3 \le x \le 9$, what is the greatest cost for order and transportation? Extreme Value Thm guarantees an absolute maximum at either $x=3$, $x=9$, or $x=4.098$.

E.V.T.

$C(3) = \$8333.33$

$C(9) = \$8611.11$

$C(4.098) = \$8213.67$

For $3 \le x \le 9$, the greatest cost is \$8611.11

Example 3: A car company introduces a new car for which the number of cars sold, S, is modeled by the function

$$S(t) = 300\left(5 - \frac{9}{t+2}\right),$$

where t is the time in months.

Use MATH 8 on CALC

a. Find the value of $S'(2.5)$. Using correct units, explain what this value represents in the context of this problem.

$\rightarrow S'(2.5) = \boxed{133.333}$

After 2.5 months, the cars are being sold at a rate of 133 cars per month.

b. Find the average rate of change of cars sold over the first 12 months. Indicate correct units of measure and explain what this value represents in the context of this problem.

$$\text{Avg. Rate of Change} = \frac{S(0)-S(12)}{0-12} = \frac{150 - 1307.143}{-12} = \boxed{96.429}$$

During the first 12 months an average of 96 cars are sold per month.

c. Is it possible that a value of c for $0 \le c \le 12$ exists such that $S'(c)$ is equal to the average rate of change? Give a reason for your answer and if such a value of c exists, find the value.

M.V.T

$\rightarrow S(t)$ is discontinuous at $t=-2$ which is not on $[0,12]$.
$S(t)$ continuous on $[0,12]$ ✓
$\rightarrow S(t)$ is diff. on $[0,12]$ ✓ (only not diff at $t=-2$)

mean value Theorem applicable

$$f'(c) = \frac{f(a)-f(b)}{a-b}$$

$$S(t) = 1500 - \frac{2700}{t+2}$$

$$S(t) = 1500 - 2700(t+2)^{-1}$$

$$S'(t) = 2700(t+2)^{-2}(1)$$

$$S'(t) = \frac{2700}{(t+2)^2}$$

$$\frac{2700}{(c+2)^2} = \frac{S(0)-S(12)}{0-12}$$

$$\frac{2700}{(c+2)^2} = \frac{150-1307.143}{-12}$$

$$-32400 = -1157.143(c+2)^2$$

$$27.999 = (c+2)^2$$

$$c+2 = 5.292$$

$$\boxed{c = 3.292}$$

increasing

Example 4:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by the equation $h(x) = f(g(x)) - 6$.

a. Find the equation of the tangent line drawn to the graph of h when $x = 3$.

$h(x) = f(g(x)) - 6$

$h(3) = f(g(3)) - 6$

$= f(4) - 6$

$= -1 - 6$

$h(3) = -7$ $(3,-7)$ point

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(3) = f'(g(3)) \cdot g'(3)$

$= f'(4) \cdot (2)$

$= 3 \cdot 2$

$h'(3) = 6$ slope

$\boxed{y+7 = 6(x-3)}$

b. Find the rate of change of h for the interval $1 < x < 3$.

Rate of change $= \dfrac{h(1)-h(3)}{1-3}$

$= \dfrac{3-(-7)}{-2} = \dfrac{10}{-2} = \boxed{-5}$

$h(1) = f(g(1)) - 6$

$= f(2) - 6$

$= 9 - 6 = 3$

$h(3) = f(g(3)) - 6$

$f(4) - 6$

$-1 - 6 = -7$

c. Explain why there must be a value of r for $1 < r < 3$ such that $h(r) = -2$.

I.V.T.

- Since f & g are differentiable then they are both continuous which means $h(x)$ is continuous on $[1, 3]$.
- $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
  $h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$ } $h(r) = -2$ is between $h(1)$ & $h(3)$

∴ The IVT guarantees there is a value r on $(1,3)$ such that $h(r) = -2$.

d. Explain why there is a value of c for $1 < c < 3$ such that $h'(c) = -5$.

M.V.T

- Since f & g are differentiable then $h(x)$ is differentiable on $(1,3)$ & continuous on $[1,3]$.

∴ The mean value Theorem guarantees a value of c on $(1,3)$ such that $h'(c) = \dfrac{h(a)-h(b)}{a-b} = -5$.