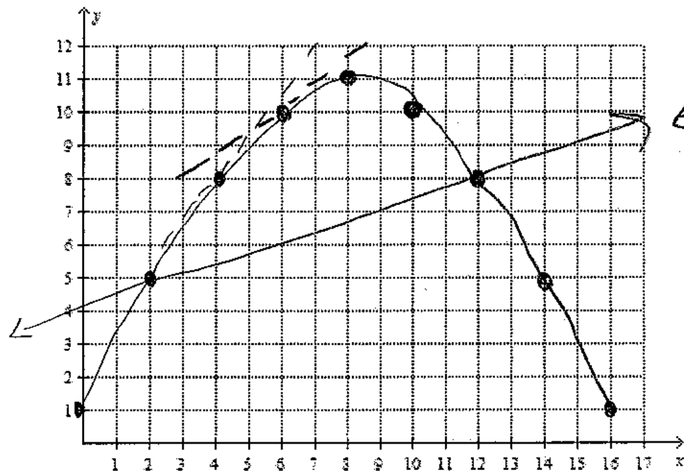
AP Calculus
Unit 5 – Applications of the Derivative – Part 2

Day 2 Notes: Mean Value Theorem & Rolle's Theorem

Consider the values of a differentiable function, $f(x)$, in the table below to answer the questions that follow. Plot the points and connect them on the grid below.

x	0	2	4	6	8	10	12	14	16
$f(x)$	1	5	8	10	11	10	8	5	1



In calculus, the derivative has many interpretations. One of the most important interpretations is that the derivative represents the Rate of Change of a Function. When speaking of rate of change, there are two rates of change that can be found that are associated with a function—average rate of change and instantaneous rate of change.

slope of secant line

slope of tangent line

Average Rate of Change of $f(x)$ on an Interval at a Point

On the interval [a,b], the Average Rate of Change is defined to be $\frac{f(a)-f(b)}{a-b}$

Instantaneous Rate of Change of $f(x)$

At a value, x=a, the instaneous rate of change is defined to be f'(a).

Find the average rate of change of $f(x)$ on the ~~at x = 4~~ interval [2, 12]. ~~Justify.~~

$$\frac{f(a)-f(b)}{a-b} = \frac{f(2)-f(12)}{2-12} = \frac{5-8}{2-12} = \frac{-3}{-10} = \boxed{\frac{3}{10}}$$

Is the instantaneous rate of change of f at $x=4$ greater than the rate of change at $x = 6$? Justify.

The inst. rate of change at x=4 is greater than at x=6 b/c the slope of the tangent at x=4 has a greater (steeper) slope.

Rolle's Theorem

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$, then there must exist at least one value, $x=c$, on (a,b) such that $f'(c)=0$.

Consider the function, $f(x)$, presented on the previous page. Does Rolle's Theorem apply on the following intervals? Explain why or why not?

Interval $[2, 14]$	$\rightarrow$ $f(x)$ is differentiable on $(2,14)$ ✓ $\rightarrow$ $f(x)$ is continuous on $[2,14]$ ✓ $\rightarrow$ $f(2)=f(14)=5$ ✓	Rolle's Thm applicable
Interval $[2, 10]$	$\rightarrow$ $f(x)$ is diff. on $(2,10)$ ✓ $\rightarrow$ $f(x)$ is cont. on $[2,10]$ ✓ $\rightarrow$ $f(2)=5$, $f(10)=10$ $f(2)\neq f(10)$	Rolle's not applicable

For each of the functions below, determine whether Rolle's Theorem is applicable or not. Then, apply the theorem to find the values of c guaranteed to exist.

1. $g(x)=9x^2-x^4$ on the interval $[-3, 0]$

- diff. on $(-3,0)$ ✓
- cont. on $[-3,0]$ ✓
- $g(-3)=9(-3)^2-(-3)^4=0$

$g(0)=9(0)^2-(0)^4=0$

$g(-3)=g(0)$ ✓

Rolle's applicable ✓

$g'(c)=0$

$18c-4c^3=0$

$2c(9-2c^2)=0$

$\cancel{c=0}$ $\quad -2c^2=-9$

$c^2=9/2$

$c=\pm\frac{3}{\sqrt{2}}$

endpt of interval

$\cancel{c=\frac{3}{\sqrt{2}}}$ $\quad \boxed{c=\frac{-3}{\sqrt{2}}}$

not in interval

2. $g(x)=\dfrac{\sin 2x}{x+2}$ on the interval $[-4, -1]$

$x\neq-2$

- $g(x)$ is discontinuous at $x=-2$ so Rolle's Theorem does not guarantee a value of c such that $g'(c)=0$

Rolle's Theorem guarantees that if a function is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there is guaranteed to exist a value of c on (a, b) where the **~~instantaneous rate of change is equal to zero.~~**

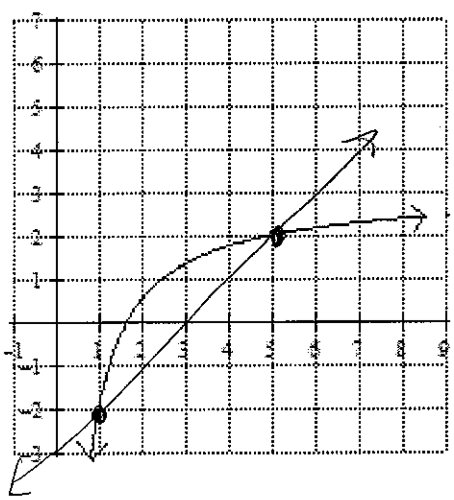
The Mean Value Theorem is similar. In fact, Rolle's Theorem is a specific case of what is known in calculus as the Mean Value Theorem.

Average
Mean Value Theorem (M.V.T.)

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there must exist a value, $x=c$, on (a,b) such that

$$f'(c) = \frac{f(a)-f(b)}{a-b}.$$

Consider the function $h(x) = 3 - \frac{5}{x}$. The graph of $h(x)$ is pictured below. Does the M.V.T. apply on the interval $[-1, 5]$? Explain why or why not. The M.V.T does not apply b/c $h(x)$ is undefined at $x=0$ which means $h(x)$ is not continuous on $[-1,5]$.



Does the M.V.T. apply on the interval $[1, 5]$? Why or why not?

- $h(x)$ is cont on $[1,5]$ ✓
- $h(x)$ is diff on $(1,5)$ ✓

} M.V.T is applicable

Graphically, what does the M.V.T. guarantee for the function on the interval $[1, 5]$? Draw this on the graph to the left.

M.V.T guarantees a value of c b/t $x=1$ and $x=5$ where the tangent slope, $h'(c)$ is the same as the secant line slope, $\frac{h(1)-h(5)}{1-5}$.

Apply the M.V.T. to find the value(s) of c guaranteed for $h(x)$ on the interval $[1, 5]$

$h(x) = 3 - 5x^{-1}$

$h'(x) = 5x^{-2}$

$h'(c) = \frac{5}{c^2}$

$\frac{5}{c^2} = \frac{h(1)-h(5)}{1-5}$

$\frac{5}{c^2} = \frac{-2-2}{1-5}$

$\frac{5}{c^2} \times \frac{-4}{-4}$

$-4c^2 = -20$

$c^2 = 5$

$c = \pm\sqrt{5}$

$c = \pm 2.236$

$\boxed{c = 2.236}$

Explain why you cannot apply the Mean Value Theorem for $f(x) = x^{\frac{2}{3}} - 2$ on the interval $[-1, 1]$.

$f(x) = \sqrt[3]{x^2} - 2$

$f'(x) = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3\sqrt[3]{x}}$

Since $f'(x)$ is undefined at $x=0$ which is on $(-1,1)$, then $f(x)$ is not differentiable on $(-1,1)$.

$\therefore$ M.V.T is not applicable for $f(x)$ on $[-1,1]$.

Find the equation of the tangent line to the graph of $f(x) = 2x + \sin x + 1$ on the interval $(0, \pi)$ at the point which is guaranteed by the mean value theorem.

$f(0) = 2(0) + \sin(0) + 1 = 1$

$f(\pi) = 2\pi + \sin\pi + 1 = 2\pi + 1$

$f'(x) = 2 + \cos x$

$f'(c) = 2 + \cos(c)$

$2 + \cos(c) = \frac{f(0) - f(\pi)}{0 - \pi}$

$2 + \cos(c) = \frac{1 - [2\pi + 1]}{0 - \pi}$

$2 + \cos(c) = \frac{-2\pi}{-\pi}$

$2 + \cos(c) = 2$

$\cos(c) = 0$

$c = \pi/2$

$f(\frac{\pi}{2}) = 2(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) + 1$

$= \pi + 1 + 1$

$= \pi + 2$

$(\frac{\pi}{2}, \pi + 2)$ point

$f'(\frac{\pi}{2}) = 2 + \cos(\frac{\pi}{2}) = 2 + 0 = 2$

S.O.T $= 2$

$\boxed{y - (\pi + 2) = 2(x - \frac{\pi}{2})}$

The Mean Value Theorem guarantees that if a function is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is guaranteed to exist a value of c where the **instantaneous rate of change at $x = c$ is equal to the average rate of change of f on the interval $[a, b]$.**