Day 6 - Composite & Inverse Functions

(Textbook Sections 1.4 & 1.5)

ESSENTIAL QUESTIONS:

1. How can composite functions be used to model real life problems?
2. How are the graphs of a function and its inverse related?

Example #1:

Let $f(x) = \sqrt{x-4}$ and $g(x) = x$. Find

$(f + g)(x) = f(x) + g(x) = \boxed{\sqrt{x-4} + x}$ $\quad x-4 \geq 0$, $x \geq 4$, domain: $[4, \infty)$

$(fg)(x) = f(x) \cdot g(x) = \boxed{x\sqrt{x-4}}$ $\quad x-4 \geq 0$, $x \geq 4$, dom: $[4, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{\sqrt{x-4}}{x}}$ $\quad x-4 \geq 0$, $x \geq 4$, $x \neq 0$, dom: $[4, \infty)$

Now state the domain of each.

Example #2:

If $f(x) = 3x^2 - x + 1$ and $h(x) = f(x - 2) - f(x)$, write the function $h(x)$ as a simplified polynomial.

$f(x-2) = 3(x-2)^2 - (x-2) + 1$
$3(x^2 - 4x + 4) - x + 2 + 1$
$3x^2 - 12x + 12 - x + 2 + 1$
$= 3x^2 - 13x + 15$

$f(x-2) - f(x) =$
$3x^2 - 13x + 15 - (3x^2 - x + 1)$
$3x^2 - 13x + 15 - 3x^2 + x - 1$
$\boxed{-12x + 14}$

DEFINITION Composition of Functions

Let f and g be two functions such that the domain of f intersects the range of g. The **composition** f **of g**, denoted $f \circ g$, is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of all x-values in the domain of g that map to $g(x)$-values in the domain of f. (See Figure 1.55.)

Example #3:

$f(x) = x^2$ and $g(x) = x - 3$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x) = f(g(x)) = f(x-3)$
$\boxed{(x-3)^2 = x^2 - 6x + 9}$

$(g \circ f)(x) = g(f(x)) = g(x^2) =$
$\boxed{x^2 - 3}$

Example #4:

$f(x) = \dfrac{1}{x-2}$ $\qquad g(x) = \sqrt{x}$

Find $f(g(x))$ and $g(f(x))$. Then state the domain of each.

$f(g(x)) = f(\sqrt{x}) = \boxed{\frac{1}{\sqrt{x} - 2}}$

dom:
$\sqrt{x} - 2 \neq 0$
$\sqrt{x} \neq 2$
$\boxed{[0,4) \cup (4, \infty)} \leftarrow x \neq 4$ & $x \geq 0$

$g(f(x)) = g\left(\frac{1}{x-2}\right) = \sqrt{\frac{1}{x-2}} = \boxed{\frac{1}{\sqrt{x-2}}}$

$\sqrt{x-2} > 0$
$x - 2 > 0 \rightarrow x > 2 \rightarrow \boxed{(2, \infty)}$

Decomposing Functions

This is the process of "undoing" the composition of functions. Look for a (nested) function inside another function. Call this g(x). The outer function will be f(x).

Example #5: For each h(x), decompose h(x) so that h(x) = f(g(x))

1. $h(x) = \sqrt{4 - x}$

$g(x) = 4 - x$, $f(x) = \sqrt{x}$

2. $h(x) = (\sin x)^2 - 2\sin x + 3$

$g(x) = \sin x$, $f(x) = x^2 - 2x + 3$

3. $h(x) = e^{\sqrt{x}}$

$g(x) = \sqrt{x}$, $f(x) = e^x$ OR
$g(x) = x$, $f(x) = e^{\sqrt{x}}$

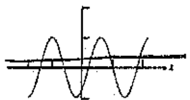
switch x & y

Inverse Functions

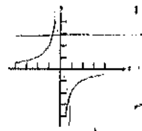
A function has an inverse if it is one-to-one.

A function is one-to-one if every x in the domain is assigned exactly one y, and every y is assigned exactly one x.

Graphically, a one-to-one function passes both the vertical line test and the horizontal line test.



NOT one-to-one
NO inv. Function



inv. function,
one-to-one

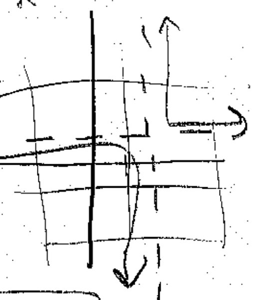
Note: The domain of the function becomes the range of the inverse. The range of the function becomes the domain of the inverse.

Example #6:

Given the function $f(x)=\frac{x+3}{x-2}$.

H.A. at $y=1$
V.A. at $x=2$

1. Is f(x) one-to-one? Justify your answer graphically.



one-to-one, passes both horiz & vert. line tests

2. What is the domain and range of f(x)?

domain: $\boxed{(-\infty,2)\cup(2,\infty)}$

Look at V.A.

range: $\boxed{(-\infty,1)\cup(1,\infty)}$

Look at H.A.

Example #6 (continued):

3. Find $f^{-1}(x)$.

$$x=\frac{y+3}{y-2}$$
$$xy-2x=y+3$$
$$xy-y=2x+3$$
$$y(x-1)=2x+3$$
$$y=\frac{2x+3}{x-1}$$
$$\boxed{f^{-1}(x)=\frac{2x+3}{x-1}}$$

HA @ $y=2$
VA @ $x=1$

4. What is the domain and range of $f^{-1}(x)$?

dom: $(-\infty,1)\cup(1,\infty)$
range: $(-\infty,2)\cup(2,\infty)$

We can prove that two functions f and g are inverses by showing that

f(g(x)) = x and g(f(x)) = x.

Example #7:

If $f(x)=\sqrt{2x-3}$ and $g(x)=\frac{x^2+3}{2}$, prove that f and g are inverses using composition.

$$f(g(x))=f\left(\frac{x^2+3}{2}\right)=\sqrt{\cancel{2}\left(\frac{x^2+3}{\cancel{2}}\right)-3}$$
$$=\sqrt{x^2+3-3}=\sqrt{x^2}=x \checkmark$$

$$g(f(x))=g(\sqrt{2x-3})=\frac{(\sqrt{2x-3})^2+3}{2}$$
$$=\frac{2x-3+3}{2}=\frac{2x}{2}=x\checkmark$$

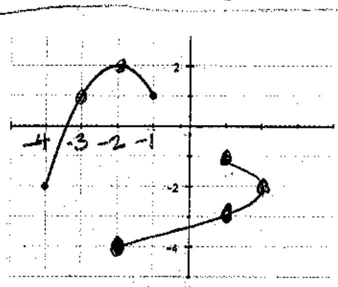
f(x) & g(x) are inverses!

Example #8:

On the same axes, sketch the graph of the INVERSE of the given function.

f(x)

x	y
-1	1
-2	2
-3	1
-4	-2



$f^{-1}(x)$

x	y
1	-1
2	-2
1	-3
-2	-4

switch x & y, then plot!

ASSIGNMENT:

pg. 124 #'s 1, 5, 14, 18, 23, 29

pg. 135 #'s 9, 11, 15, 17, 21, 24, 25, 29

TEST ON THURSDAY!!