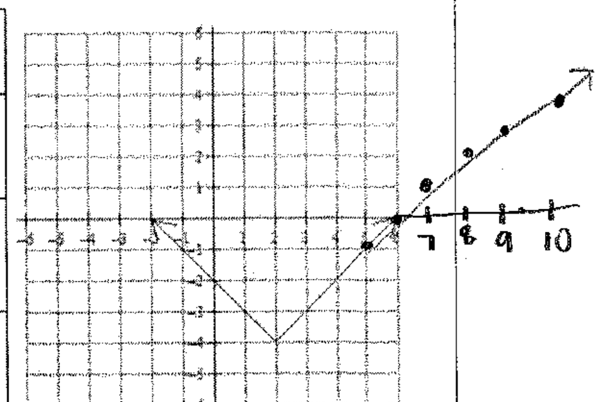
AP Calculus
Unit 1 - Limits & Continuity

Day 2: Analytical Approach to Finding Limits

Consider the function, $f(x)=\frac{1}{2}\left|-2x+4\right|-4$, for a moment. The graph of $f(x)$ is pictured below. From the graph, determine the following limits.

$\lim\limits_{x\to a} f(x)$	**Find $f(a)$ using the equation.**	**Find $\lim\limits_{x\to a} f(x)$ from the graph.**
$\lim\limits_{x\to 0} f(x)$	$f(0)=$ $\frac{1}{2}\left|-2(0)+4\right|-4$ $\frac{1}{2}(4)-4=-2$	-2
$\lim\limits_{x\to 2^+} f(x)$	$f(2)=$ $\frac{1}{2}\left|-2(2)+4\right|-4$ $\frac{1}{2}(0)-4=-4$	-4
$\lim\limits_{x\to 10} f(x)$	$f(10)=$ $\frac{1}{2}\left|-2(10)+4\right|-4$ $\frac{1}{2}(+16)-4=4$	4



When a function is defined and continuous at a value, $x=a$, how can $\lim\limits_{x\to a} f(x)$ be found analytically?

Provided that $f(x)$ is continuous at $x=a$,

$$\lim_{x\to a} f(x)=f(a).$$

Find each of the following limits analytically. *DIRECT SUBSTITUTION*

a) $\lim_{x\to 3} \frac{1}{2}x^2 - 2x + 3$

$\frac{1}{2}(3)^2 - 2(3) + 3$

$4.5 - 6 + 3 = \boxed{1.5}$

b. $\lim_{x\to 3} \frac{5x+2}{2x-3} = \frac{5(3)+2}{2(3)-3} = \boxed{\frac{17}{3}}$

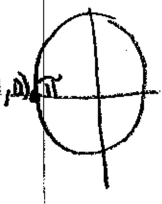
c. $\lim_{x\to 2^-} \frac{\sqrt{x+2}-1}{x+1} = \frac{\sqrt{2+2}-1}{2+1}$

$= \frac{\sqrt{4}-1}{3} = \boxed{\frac{1}{3}}$

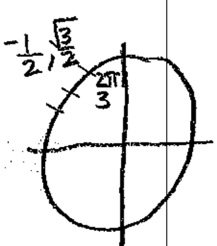
d. $\lim_{\theta\to \frac{\pi}{2}} \sin 2\theta = \sin\left[2\left(\frac{\pi}{2}\right)\right]$

$= \sin\pi$

$= \boxed{0}$



e. $\lim_{\theta\to \frac{2\pi}{3}} \frac{\cos\theta}{\theta}$



$\frac{\cos\left(\frac{2\pi}{3}\right)}{\frac{2\pi}{3}} = \frac{-\frac{1}{2}}{\frac{2\pi}{3}}$

$= -\frac{1}{2} \cdot \frac{3}{2\pi} = \boxed{\frac{-3}{4\pi}}$

f. $\lim_{x\to 9} \log_8(11-x)$

$= \log_8(11-9)$

$= \log_8(2)$

$\log_8(2) = x$

$8^x = 2$

$8^x = 8^{1/3}$ $\quad x = \boxed{1/3}$

Analytically Finding Limits of Functions at Undefined Values

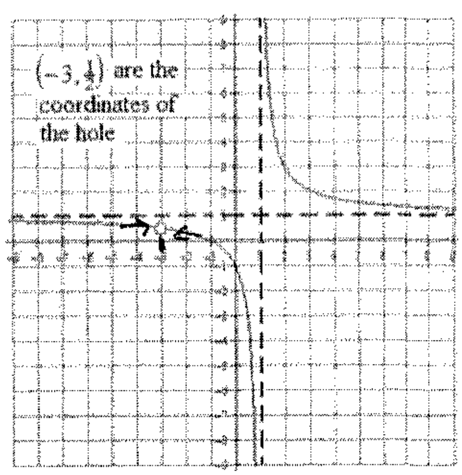
What happens if we try to evaluate the limits below by the direct substitution method that was used in the previous six examples?

$$\lim_{x\to -3} \frac{x^2+4x+3}{x^2+2x-3} \qquad \lim_{x\to 1^-} \frac{x^2+4x+3}{x^2+2x-3} \qquad \lim_{x\to 1^+} \frac{x^2+4x+3}{x^2+2x-3}$$

$\frac{(-3)^2+4(-3)+3}{(-3)^2+2(-3)-3} = \frac{0}{0}$ undefined

$\frac{(1)^2+4(1)+3}{(1)^2+2(1)-3} = \frac{8}{0}$ undefined

Just because a function is undefined at a value of x does not mean that a conclusion cannot be reached about the limit. Consider the rational function above. From the graph of the function pictured to the right, what is the value of each limit below?



$\lim_{x\to -3} \frac{x^2+4x+3}{x^2+2x-3} = \boxed{1/2}$

left $\lim_{x\to 1^-} \frac{x^2+4x+3}{x^2+2x-3} = \boxed{-\infty}$

Right $\lim_{x\to 1^+} \frac{x^2+4x+3}{x^2+2x-3} = \boxed{\infty}$

The task now is to determine how to find these limits analytically. How was it that we found the discontinuities of a rational function in pre-calculus?

① Factor the numerator & denominator.
② If a factor canceled, the function had a point of discontinuity when the factor = 0.
③ If a factor from the denominator did not cancel, the function had a non-removable infinite discontinuity

We will perform the same algebraic analysis to find the limit of the removable, point discontinuities. Let's do this **Cancellation Process** below.

$$\lim_{x\to -3} \frac{x^2+4x+3}{x^2+2x-3}$$

① Factor numerator & denominator.

$$\frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-1)}$$

② $\lim\limits_{x\to -3} \frac{(x+1)}{(x-1)} = \frac{-3+1}{-3-1} = \frac{-2}{-4} = \boxed{\frac{1}{2}}$

now do direct substitution

Based on our knowledge from pre-calculus, we know that if a rational function has a non-removable infinite discontinuity, graphically a vertical asymptote exists. Since the y – values do not approach one specific value from both sides at a vertical asymptote, then the limit does not exist. However, we can determine if the one sided limits approach $-\infty$ or ∞. In order to do this analytically, we will marry the numerical, graphical, and algebraic approaches. For each limit below, determine the sign of the simplified function at the value to the right or the left of $x = 1$.

LEFT $\lim\limits_{x\to 1^-} \frac{x^2+4x+3}{x^2+2x-3} = \lim\limits_{x\to 1^-} \frac{(x+1)}{(x-1)}$ Factored

RIGHT $\lim\limits_{x\to 1^+} \frac{x^2+4x+3}{x^2+2x-3}$

Value of x to the left of $x=1$	Simplified function $\frac{x+1}{x-1}$
0.9	$\frac{.9+1}{.9-1} = \frac{+}{-}$ = neg

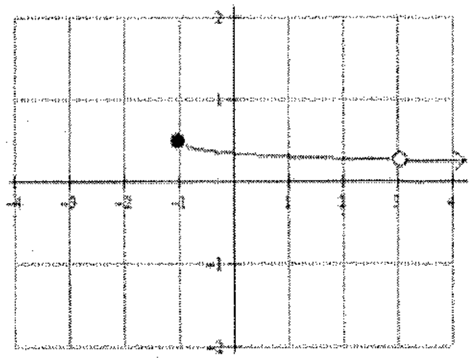
Value of x to the right of $x=1$	Simplified function $\frac{x+1}{x-1}$
1.1	$\frac{1.1+1}{1.1-1} = \frac{+}{+}$ = pos.

$\lim\limits_{x\to 1^-} \frac{x^2+4x+3}{x^2+2x-3} = \boxed{-\infty}$

$\lim\limits_{x\to 1^+} \frac{x^2+4x+3}{x^2+2x-3} = \boxed{\infty}$

The graph of a function $g(x) = \frac{\sqrt{x+1}-2}{x-3}$ is pictured to the right. Often, **rationalization** can be used to evaluate a limit analytically. Find the following limit.

$$\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3}$$



* can't use direct substitution b/c you will get 0 in the denom.

$$\frac{(\sqrt{x+1}-2)}{x-3} \cdot \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)} = \frac{x+1+\cancel{2\sqrt{x+1}}-\cancel{2\sqrt{x+1}}-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}$$

$$\lim_{x \to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

Write the equation of the piece-wise defined function pictured to the right.

Line $y=mx+b$

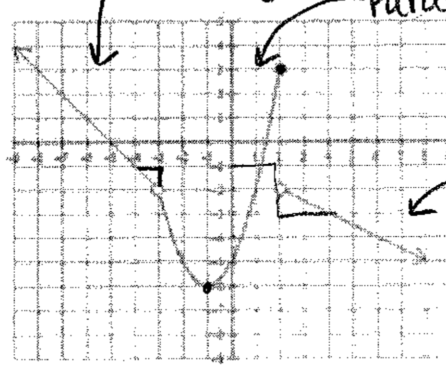
$y=-x-5$

Parabola → down 6 & left 1

$y=(x+1)^2-6$

Line $y=mx+b$

$y=-\frac{1}{2}x-1$



	Equation of Each Piece	Constraint of Each Piece
$f(x)=$	$-x-5$	$x<-3$
	$(x+1)^2-6$	$-3<x\le 2$
	$-\frac{1}{2}x-1$	$x>2$

Use the equation that you just wrote to find each of the following limits. Confirm your results based on the graph. If a limit does not exist, state why.

a) $\lim_{x \to 2^+} f(x) = \boxed{-2}$

Right

b) $\lim_{x \to 2} f(x)$ D.N.E. $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

c) $\lim_{x \to -3^-} f(x) = \boxed{-2}$

Left

d) $\lim_{x \to -3^+} f(x) = \boxed{-2}$

Right

Consider the function below to find each limit. If a limit does not exist, state why.

$$G(x) = \begin{cases} 2x^2 + 3x, & x < -2 \\ -\frac{1}{2}x + 1, & x > -2 \end{cases}$$

a) $\lim_{x \to -2^{-}} G(x)$ $2(-2)^2 + 3(-2)$

$2(4) - 6 = \boxed{2}$

Left $x < -2$

b) $\lim_{x \to -2^{+}} G(x)$ $-\frac{1}{2}(-2) + 1 = 1 + 1 = \boxed{2}$

Right $x > -2$

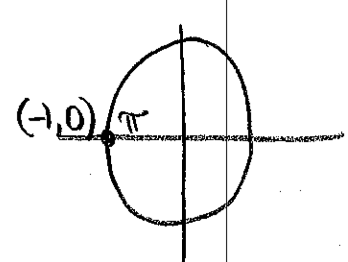
c) $\lim_{x \to -2} G(x) = \boxed{2}$ since $\lim_{x \to -2^-} G(x) = \lim_{x \to -2^+} G(x)$

Find each of the following limits analytically. Show your algebraic analysis.

a. $\lim_{x \to e} \frac{\ln x}{2x} = \frac{\ln(e)}{2(e)} = \boxed{\frac{1}{2e}}$

b. $\lim_{x \to 5^{-}} \left(\frac{2}{5}x^2 + 2x\right) = \frac{2}{5}(5)^2 + 2(5)$

$10 + 10 = \boxed{20}$



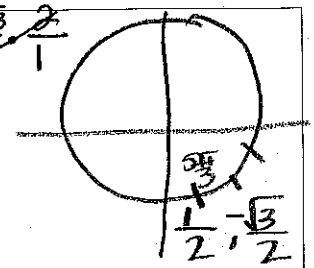
c. $\lim_{\theta \to \pi} \left(\sin^2\theta + 2\cos\theta\right) = \sin^2(\pi) + 2\cos(\pi)$

$= (\sin\pi)^2 + 2(\cos\pi)$

$= (0)^2 + 2(-1)$

$= \boxed{-2}$

★ d. $\lim_{\alpha\to\frac{5\pi}{3}} \frac{\tan\alpha}{\alpha^2} = \frac{\tan\left(\frac{5\pi}{3}\right)}{\left(\frac{5\pi}{3}\right)^2} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = \frac{-\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{1}$



↓

$= \frac{-\sqrt{3}}{\frac{25\pi^2}{9}} = \frac{-\sqrt{3}}{1}\cdot\frac{9}{25\pi^2} = \boxed{\frac{-9\sqrt{3}}{25\pi^2}}$

★ e. $\lim_{x\to-2} \frac{x^2-x-6}{2x+4} = \frac{(-2)^2-(-2)-6}{2(-2)+4} = \frac{\quad}{0}$

↓ FACTOR!

$\frac{(x-3)\cancel{(x+2)}}{2\cancel{(x+2)}} \longrightarrow \lim_{x\to-2}\frac{(x-3)}{2} = \frac{-2-3}{2} = \boxed{\frac{-5}{2}}$

f. $\lim_{x\to3} \frac{x+5}{x^2-9} \rightarrow$ Factor! $\lim_{x\to3} \frac{(x+5)}{(x+3)(x-3)}$

$\cancel{(3)^2-9}=0$

Vertical asymptotes

$\boxed{\text{D.N.E.}}$

$a=2x \quad b=3$

g. $\lim_{x\to\frac{3}{2}} \frac{8x^3-27}{2x-3} \rightarrow$ Factor $\frac{\cancel{(2x-3)}(4x^2+6x+9)}{\cancel{(2x-3)}}$

$\cancel{2\left(\frac{3}{2}\right)-3}$

$\lim_{x\to\frac{3}{2}} (4x^2+6x+9) = 4\left(\frac{3}{2}\right)^2+6\left(\frac{3}{2}\right)+9$

$\cancel{4}\left(\frac{9}{\cancel{4}}\right)+9+9 = \boxed{27}$

h. $\lim_{x\to -2} \frac{\sqrt{2x+5}-1}{x+2}$ $\cancel{-2+2}$

$$\frac{(\sqrt{2x+5}-1)}{x+2}\cdot\frac{(\sqrt{2x+5}+1)}{(\sqrt{2x+5}+1)} = \frac{2x+5+\sqrt{2x+5}-\sqrt{2x+5}-1}{(x+2)(\sqrt{2x+5}+1)}$$

$$= \frac{2x+4}{(x+2)(\sqrt{2x+5}+1)} = \frac{2\cancel{(x+2)}}{\cancel{(x+2)}(\sqrt{2x+5}+1)}$$

$$= \frac{2}{\sqrt{2x+5}+1}$$

$$\lim_{x\to -2}\frac{2}{\sqrt{2x+5}+1} = \frac{2}{\sqrt{2(-2)+5}+1} = \frac{2}{2} = \boxed{1}$$

i. $\lim_{x\to 1}\frac{1-\sqrt{2x^2-1}}{x-1}$

$$\frac{(1-\sqrt{2x^2-1})}{x-1}\cdot\frac{(1+\sqrt{2x^2-1})}{(1+\sqrt{2x^2-1})} = \frac{1-(2x^2-1)}{(x-1)(1+\sqrt{2x^2-1})} = \frac{-2x^2+2}{(x-1)(1+\sqrt{2x^2-1})}$$

$$= \frac{-2(x^2-1)}{(x-1)(1+\sqrt{2x^2-1})}$$

$$= \frac{-2(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(1+\sqrt{2x^2-1})}$$

$$\lim_{x\to 1}\frac{-2(x+1)}{(1+\sqrt{2x^2-1})} = \frac{-2(1+1)}{1+\sqrt{2(1)^2-1}} = \frac{-4}{2} = \boxed{-2}$$

j. $\lim_{x\to 0}\frac{\frac{1}{x+2}+\frac{1}{x}}{x}$

$$\frac{1}{x+2}\cdot^{(x)}_{(x)} + \frac{1}{x}\cdot^{(x+2)}_{(x+2)} = \frac{x}{x(x+2)} + \frac{x+2}{x(x+2)} = \frac{2x+2}{x(x+2)}$$

$$= \frac{2(x+1)}{x(x+2)}$$

$$\frac{2(x+1)}{x(x+2)}\cdot\frac{1}{x} = \frac{2(x+1)}{x^2(x+2)}$$

$$\lim_{x\to 0}\frac{2(x+1)}{x^2(x+2)} = \boxed{\text{D.N.E}}$$

The factor of x would not cancel through algebraic means. Thus, at x=0 a vertical asymptote exists, meaning the limit doesn't exist.

k. $\lim_{x\to 2^+}\frac{3x^2+7x+2}{x^2-4} = \frac{(3x+1)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$

$\frac{6}{6 \mid 1}$

$3x^2+6x+1x+2$

$3x(x+2)+1(x+2)$

$$\lim_{x\to 2^+}\frac{(3x+1)}{(x-2)}$$

plug in 2.1

$$\frac{3(2.1)+1}{2.1-1} = \frac{+}{+} = \text{pos}$$

V.A. check using signs

$$= \boxed{\infty}$$

can't factor!

l. $\lim_{x\to 3^+} \frac{2x+5}{x-3}$ $= \frac{2(3.1)+5}{3.1-3} = \frac{+}{+} = \text{pos}$

V.A.
plug in 3.1

$\boxed{\infty}$

m. $\lim_{x\to 3^-} \frac{2x+5}{x-3}$ $= \frac{2(2.9)+5}{2.9-3} = \frac{+}{-} = \text{neg}$

V.A.
plug in 2.9

$= \boxed{-\infty}$

If $f(x) = 2x^2 - 3x + 4$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

lim $2(x+h)^2 - 3(x+h) + 4 - (2x^2 - 3x + 4)$

$2(x^2+2xh+h^2) - 3x - 3h + 4$

$\cancel{2x^2} + 4xh + 2h^2 \cancel{-3x} - 3h \cancel{+4} \cancel{-2x^2} \cancel{+3x} \cancel{-4}$

$\frac{4xh + 2h^2 - 3h}{h}$

$= \frac{\cancel{h}(4x+2h-3)}{\cancel{h}} \rightarrow \lim_{h\to 0}(4x+2h-3) = 4x + 2(0) - 3$

$= \boxed{4x-3}$

Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Find each of the following limits in terms of L and M.

1. $\lim_{x\to a}[f(x)+g(x)]$ $= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ $\boxed{L+M}$

2. $\lim_{x\to a}[f(x)-g(x)]$ $\lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ $\boxed{L-M}$

3. $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \boxed{\dfrac{L}{M}}$

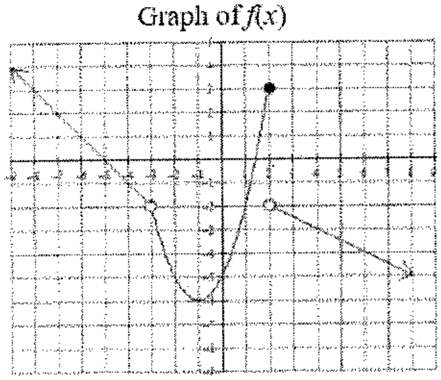
4. $\lim_{x\to a}[f(x)\cdot g(x)]$ $\lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = \boxed{L \cdot M}$

5. $\lim_{x\to a}[c\cdot f(x)]$ $= \lim_{x\to a} c \cdot \lim_{x\to a} f(x)$ $= \boxed{c \cdot L}$

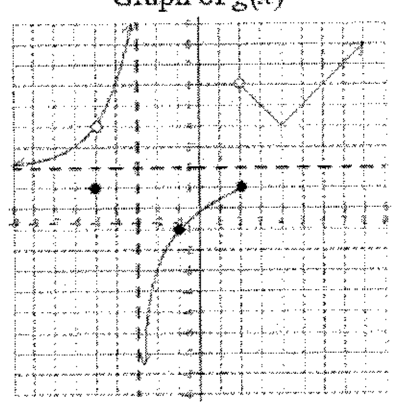
6. $\lim_{x\to a}[f(x)]^p$ $= \left[\lim_{x\to a} f(x)\right]^p$ $= \boxed{L^p}$

7. $\lim_{x\to a} c$ $= \boxed{c}$

Graph of $f(x)$



Graph of $g(x)$



Find each of the following limits applying the properties of limits. If a limit does not exist, state why.

$\lim_{x\to 2^-}[f(x)+g(x)]$

$\lim_{x\to 2^-} f(x) = 3$ $\lim_{x\to 2^-} g(x) = 1$

$3+1 = \boxed{4}$

$\lim_{x\to 2^+}\sqrt{2g(x)}$

$\lim_{x\to 2^+} g(x) = 6$ $\sqrt{2(6)} = \sqrt{12} = \boxed{2\sqrt{3}}$

$\lim_{x\to -3}[f(x)-g(x)]$

$\lim_{x\to -3} f(x) = -2$ $\lim_{x\to -3} g(x) = \text{D.N.E.}$

$\boxed{\text{D.N.E}}$ b/c $\lim_{x\to -3} g(x)$ DNE (vert. asymp)

$\lim_{x\to 6}\dfrac{-2f(x)}{g(x)}$

$\lim_{x\to 6} f(x) = -4$

$\lim_{x\to 6} g(x) = 6$

$\dfrac{-2(-4)}{6} = \dfrac{8}{6} = \boxed{\dfrac{4}{3}}$