AP Calculus
Unit 1 - Limits & Continuity

Day 1: Numerical & Graphical Approach to Finding Limits

Rational Function

The equation of the function graphed to the right is

$f(x)=\dfrac{2x^2+5x-3}{x^2-9}$. The coordinates of the hole in the graph are $\left(-3,\frac{7}{6}\right)$.

Horizontal asymptote at $y=\frac{2}{1} \rightarrow y=2$

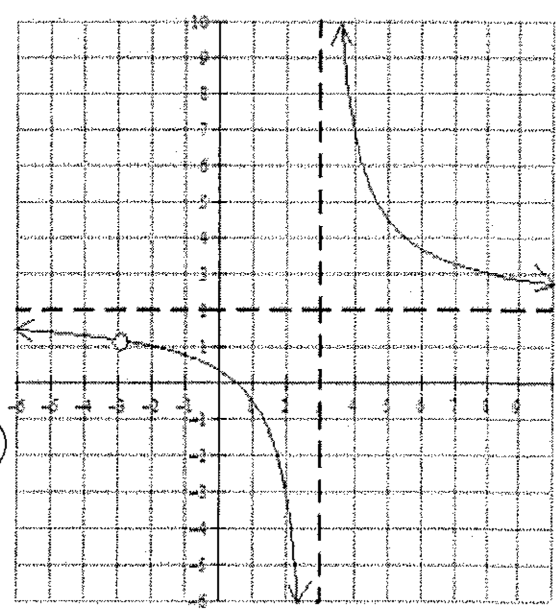
$f(x)=\dfrac{(2x-1)(x+3)}{(x+3)(x-3)}$ ← vertical asymptote at $x=3$

↑ hole at $x=-3$

$\dfrac{2(-3)-1}{-3-3}=\dfrac{-7}{-6}=\dfrac{7}{6}$

hole at $\left(-3,\frac{7}{6}\right)$

$(2x^2+6x)(-1x-3)$
$2x(x+3)-1(x+3)$
$(2x-1)(x+3)$



Pre-calculus Statements	Calculus Limit Notation
As $x \to -\infty$, the graph of $f(x) \to$ 2.	$\lim_{x \to -\infty} f(x) = 2$
As $x \to \infty$, the graph of $f(x) \to$ 2.	$\lim_{x \to \infty} f(x) = 2$
As $x \to -3$ from the left, the graph of $f(x) \to$ 7/6.	$\lim_{x \to -3^-} f(x) = \frac{7}{6}$
As $x \to -3$ from the right, the graph of $f(x) \to$ 7/6.	$\lim_{x \to -3^+} f(x) = 7/6$
As $x \to 3$ from the left, the graph of $f(x) \to$ $-\infty$.	$\lim_{x \to 3^-} f(x) = -\infty$
As $x \to 3$ from the right, the graph of $f(x) \to$ ______.	$\lim_{x \to 3^+} f(x) = \infty$

Based on what you have just seen, how might you informally define what the value of a limit represents in terms of the graph?

$\lim_{x \to a} f(x)$ is the y-value the graph approaches as x approaches a.

A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \dfrac{2x^2 + 5x - 3}{x^2 - 9}$ — same rational function
- Complete the table to below to perform a numerical analysis of the function as $x \to -\infty$ and ∞.

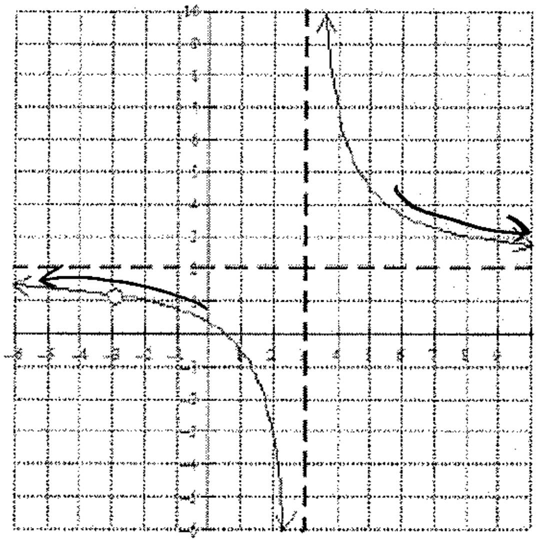
x	−1000	−500	−100	−50	50	100	500	1000
f(x)	1.995	1.990	1.951	1.906	2.106	2.052	2.010	2.005

Based on the values in the table, what are the values of

$$\lim_{x \to -\infty} \frac{2x^2 + 5x - 3}{x^2 - 9} = 2 \qquad \lim_{x \to \infty} \frac{2x^2 + 5x - 3}{x^2 - 9} = 2$$

How is the numerical analysis above related in the graph of the function pictured below?

As $x \to -\infty$, the graph of f(x) gets infinitely close to y=2 through values that are less than 2.



As $x \to \infty$, the graph of f(x) gets infinitely close to y=2 through values that are greater than 2.

A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \dfrac{2x^2 + 5x - 3}{x^2 - 9}$
- Complete the table to below to perform a numerical analysis of the function as $x \to -3$ from the left and from the right.

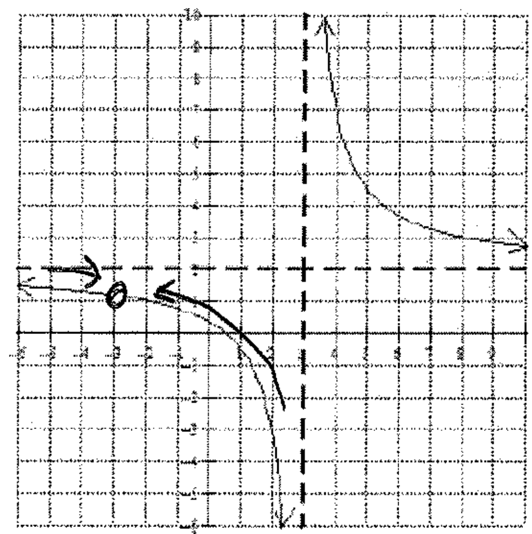
x	−3.75	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9	−2.75
$f(x)$	1.259	1.180	1.168	1.167	1.167	1.165	1.153	1.130

Based on the values in the table, what are the values of

$\lim_{x \to -3^-} \dfrac{2x^2+5x-3}{x^2-9} = 1.167$ (Left)

$\lim_{x \to -3^+} \dfrac{2x^2+5x-3}{x^2-9} = 1.167$ (Right)

How is the numerical analysis above related in the graph of the function pictured below?



As x values approach −3 from the left & Right, y-values approach 1.1167 or 7/6 which is the y-value of the point of discontinuity at x=−3.

A Numerical Analysis of Limits

- Now let's consider the function $f(x) = \dfrac{2x^2 + 5x - 3}{x^2 - 9}$
- Complete the table to below to perform a numerical analysis of the function as $x \to 3$ from the left and from the right.

(3)

x	2.75	2.9	2.99	2.999	3.001	3.01	3.1	3.75
f(x)	-18	-48	-498	-4998	5002	502	52	8.1667

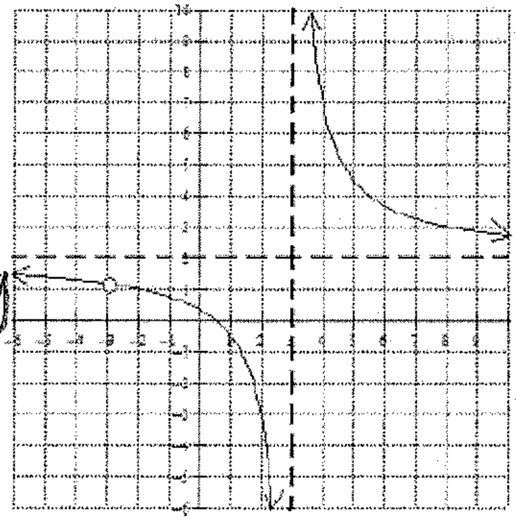
Based on the values in the table, what are the values of

$$\lim_{x \to 3^-} \frac{2x^2 + 5x - 3}{x^2 - 9} = -\infty \qquad \lim_{x \to 3^+} \frac{2x^2 + 5x - 3}{x^2 - 9} = \infty$$

left Right

How is the numerical analysis above related in the graph of the function pictured below?

As $x \to 3$ from the left, the graph is approaching $-\infty$ so the values of y to the left should be very negative.



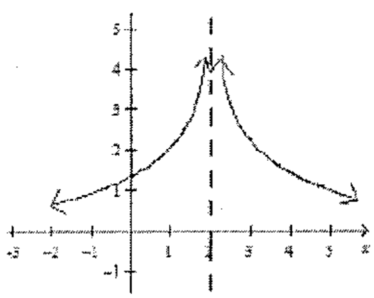
As $x \to 3$ from the right, the graph is approaching $+\infty$ so the y values to the right should be very positive.

Limit Existence Theorem

$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = b$ where b is any real number.

Limits That Do Not Exist

Example #1



Find each of the following from the graph.

a) $\lim_{x\to 2^-} f(x) =$ ∞

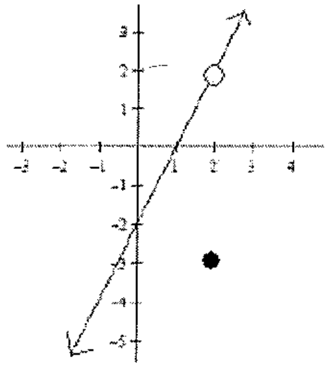
b. $\lim_{x\to 2^+} f(x) =$ ∞

c) $f(2) =$ undefined (asymptote)

d) Does $\lim_{x\to 2} f(x)$ exist or not? Why or why not?

NO, $\lim_{x\to 2} f(x)$ D.N.E because although $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, the limit is not a real #.

Example #2



Find each of the following from the graph.

a) $\lim_{x\to 2^-} f(x) =$ **2**

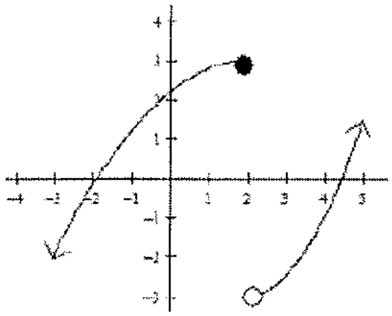
b. $\lim_{x\to 2^+} f(x) =$ **2**

c) $f(2) =$ **-3**

d) Does $\lim_{x\to 2} f(x)$ exist or not? Why or why not?

Yes, $\lim_{x\to 2} f(x)$ exists. $\lim_{x\to 2} f(x) = 2$

Example #3



Find each of the following from the graph.

a) $\lim_{x\to 2^-} f(x) =$ **3**

b. $\lim_{x\to 2^+} f(x) =$ **-3**

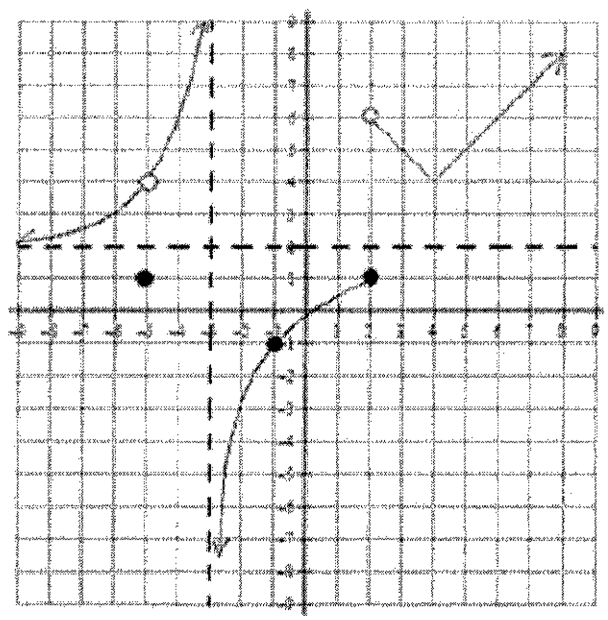
c) $f(2) =$ **3**

d) Does $\lim_{x\to 2} f(x)$ exist or not? Why or why not?

NO, $\lim_{x\to 2} f(x)$ D.N.E b/c $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$.

A Graphical Analysis of Limits

- Consider the graph of the function, f(x), graphed below.



A Graphical Analysis of Limits

Using the graph, find the value of each of the following limits. If a limit does not exist, state why.

a) $\lim_{x \to -3^-} f(x) = \boxed{\infty}$ left

b) $\lim_{x \to -5} f(x) = \boxed{4}$

c) $\lim_{x \to -1} f(x) = \boxed{-1}$

c) $\lim_{x \to -3} f(x) = \boxed{D.N.E}$

$\lim_{x \to -3^-} f(x) \neq \lim_{x \to -3^+} f(x)$

d) $\lim_{x \to 2^-} f(x) = \boxed{1}$ left

e) $\lim_{x \to 2^+} f(x) = \boxed{6}$

f) $\lim_{x \to 2} f(x) = \boxed{D.N.E}$

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

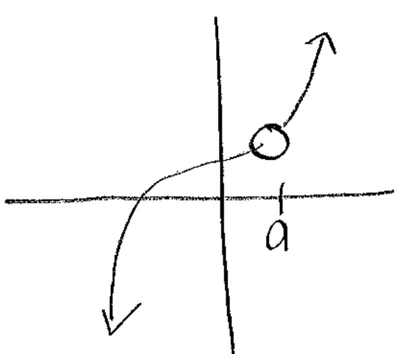
g) $\lim_{x \to -\infty} f(x) = \boxed{2}$

h) $\lim_{x \to \infty} f(x) = \boxed{\infty}$

Your Turn:

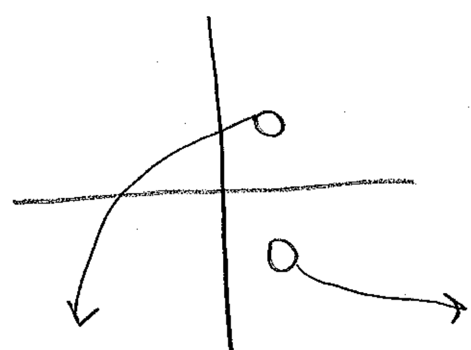
Based on what you have seen so far, does $f(a)$ have to be defined in order for the $\lim_{x \to a} f(x)$ to exist? Draw and explain two different graphs to justify your reasoning. In both graphs, $f(a)$ should be undefined but in one graph, the limit should exist while in the second one, it should not exist.

HOLE



$f(a)$ is undefined but $\lim_{x \to a} f(x)$ exists.

Jump



$f(a)$ is undefined and $\lim_{x \to a} f(x)$ does not exist.

Your Turn:

Below is a table of values of an exponential function. Use the table to find the limits that follow.

x	-9	-5	-3	-1	1	3	9
$f(x)$	513	33	9	3	1.5	1.125	1.002

a) $\lim_{x\to-\infty} f(x)$ = ∞

b) $\lim_{x\to-3} f(x)$ = 9

c) $\lim_{x\to1} f(x)$ = 1.5

d) $\lim_{x\to\infty} f(x)$ = 1

Your Turn:

Below is a table of values of a rational function. Use the table to find the limits that follow.

x	-1000	-1.001	-1	-0.999	0	1.999	2	2.001	1000
$f(x)$	1.002	2001	Undefined	-1999	-1	0.333	Undefined	0.334	0.998

a) $\lim_{x\to-\infty} f(x)$ = 1

b) $\lim_{x\to-1^-} f(x)$ = ∞
left

c) $\lim_{x\to-1^+} f(x)$ = $-\infty$
Right

d) $\lim_{x\to2} f(x)$ = 1/3

e) $\lim_{x\to\infty} f(x)$ = 1

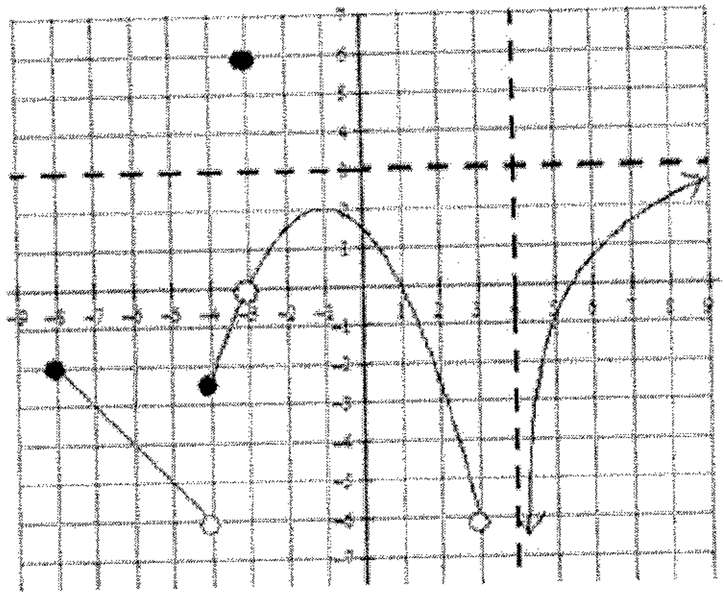
f) $\lim_{x\to-1} f(x)$ = D.N.E

b/c $\lim_{x\to-1^-} f(x) \neq \lim_{x\to-1^+} f(x)$

Your Turn:

A Graphical Analysis of Limits

Consider the graph of the function, g(x), graphed below.



Your Turn:

A Graphical Analysis of Limits

Find the value of each of the following limits using the graph of g(x). If a limit does not exist, state why.

a) $\lim_{x \to -3^-} g(x) = \boxed{0}$ left

b) $\lim_{x \to -6} g(x) = \boxed{-4}$

c) $\lim_{x \to -1^+} g(x) = \boxed{2}$ right

d) $\lim_{x \to -3^+} g(x) = \boxed{0}$

e) $\lim_{x \to 4^-} g(x)$ D.N.E

g(x) does not approach 4 from the left

f) $\lim_{x \to 4^+} g(x) = \boxed{-\infty}$

g) $\lim_{x \to 4} g(x)$ D.N.E

$\lim_{x \to 4^-} g(x) \neq \lim_{x \to 4^+} g(x)$

h) $\lim_{x \to -4^+} g(x) = \boxed{-2.5}$

i) $\lim_{x \to -4^-} g(x) = \boxed{-6}$