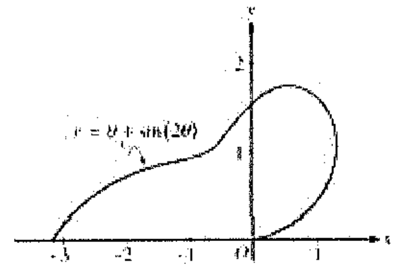
AP Calculus BC
Unit 11 – Parametric Equations & Polar Coordinates

Day 8 Notes: Polar Graphs & Area (Day 2)

2005 BC Exam - #2 – Calculator Active

r = meters   θ = radians



The curve above is drawn in the xy-plane ~~and is~~ described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the $\boxed{\text{area}}$ bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) $\text{Area} = \frac{1}{2}\int_0^{\pi} (\theta + \sin 2\theta)^2 \, d\theta = \boxed{4.782}$

(b) $x = r\cos\theta$

$x = (\theta + \sin 2\theta)(\cos\theta)$

Function Mode

$\underbrace{(\theta + \sin 2\theta)(\cos\theta)}_{y_1} = \underbrace{-2}_{y_2}$

Find intersection b/t $0 \le \theta < \pi$

$\boxed{\theta = 2.786}$

(c) $\frac{\pi}{3} < \theta < \frac{2\pi}{3} \rightarrow \frac{dr}{d\theta}$ is negative. (Derivative is negative)

If $\frac{dr}{d\theta}$ is negative, then r is decreasing on $(\pi/3, 2\pi/3)$. Therefore, the distance from the curve to the origin is decreasing.

(d) Greatest distance from the origin = MAX on $[0, \pi/2]$

$$\underbrace{1 + 2\cos(2\theta)}_{y_1} = \underbrace{0}_{y_2} \leftarrow \frac{dr}{d\theta} = 0$$

Find intersection → OR

$\theta = 1.047$

$1\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\theta = \frac{2\pi}{6}, \frac{4\pi}{6}$

$\frac{\pi}{3}, \frac{2\pi}{3}$

	θ	$r = \theta + \sin 2\theta$
endpt. →	0	0
$\frac{\pi}{3}$	1.047	1.913
endpt. →	$\pi/2$	1.571

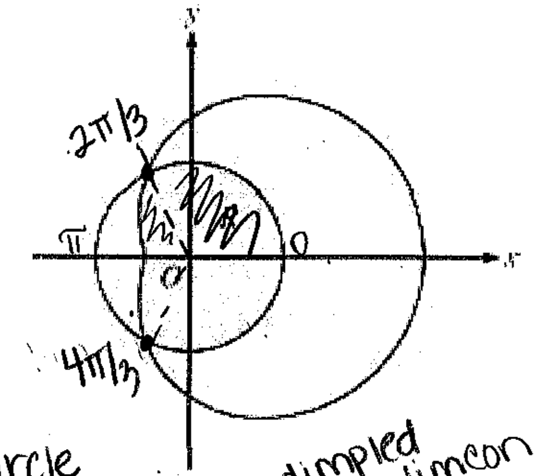
→ $r = 0 + \sin(0) = 0$

→ $r = 1.047 + \sin(2 \cdot 1.047) = 1.913$

→ $r = \pi/2 + \sin(2 \cdot \pi/2) = 1.571$

$\boxed{\text{The greatest distance occurs at } \theta = 1.047. \text{ or } \pi/3}$

2007 BC Exam - #3 – Calculator Active



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $\boxed{r = 3 + 2\cos\theta}$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) $A = \frac{1}{2}\int_0^{2\pi/3} (2)^2\, d\theta = 4.189$

$A = \frac{1}{2}\int_{2\pi/3}^{\pi} (3+2\cos\theta)^2\, d\theta = 0.996$

$4.189 + 0.996 = 5.185$

$5.185(2) = \boxed{10.370}$

(b) $\frac{dr}{dt} = \frac{dr}{d\theta} = -2\sin\theta \rightarrow$ at $\theta = \pi/3$, $\frac{dr}{dt} = -2\sin(\pi/3) = \boxed{-1.732}$

$r = 3 + 2\cos\theta$

$r = 3 + 2\cos(\pi/3)$
$= 4$

Since $\frac{dr}{dt} < 0$ and $r > 0$, this means $r = 3 + 2\cos\theta$ at $\theta = \pi/3$ is moving towards the origin at this rate.
(distance is decreasing)

ⓒ $r = 3 + 2\cos\theta$

$$\frac{dy}{dt} = \frac{dy}{d\theta}$$

$3\sin(\pi/3) + 2\cos(\pi/3)\sin(\pi/3)$
$= 3.464$

$y = r\sin\theta$

$y = (3 + 2\cos\theta)(\sin\theta) = 3\sin\theta + 2\cos\theta\sin\theta$

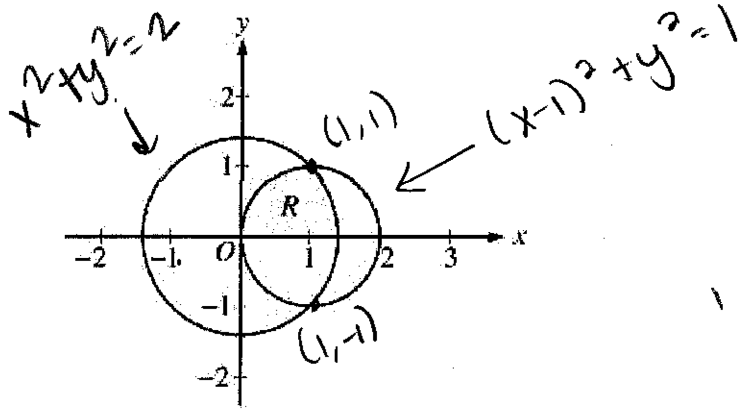
$$\frac{dy}{d\theta} = 3\cos\theta + (-2\sin\theta)(\sin\theta) + (2\cos\theta)(\cos\theta)$$

$$\frac{dy}{d\theta} = \frac{dy}{dt} = 3\cos\theta - 2\sin^2\theta + 2\cos^2\theta$$

at $\theta = \pi/3 \rightarrow 3\cos(\pi/3) - 2\sin^2(\pi/3) + 2\cos^2(\pi/3) = \boxed{1/2}$

Since $\frac{dy}{dt} > 0$ and $y > 0$, this means $r = 3 + \cos\theta$ at $\theta = \pi/3$ is moving up (or away) from the x-axis.

2003 BC Exam (Form B) - #2 - No Calculator



The figure above shows the graphs of the circles $\boxed{x^2 + y^2 = 2}$ and $\boxed{(x-1)^2 + y^2 = 1.}$ The graphs intersect at the points $(1, 1)$ and $(1, -1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.

(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

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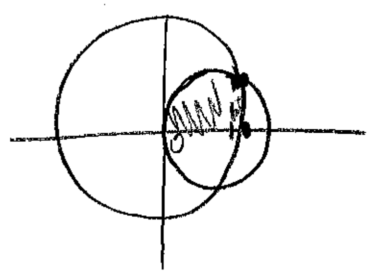
respect to x → solve for y

(a)

$x^2+y^2=2$ $\qquad$ $(x-1)^2+y^2=1$

$y^2=2-x^2$ $\qquad$ $y^2=1-(x-1)^2$

$y=\pm\sqrt{2-x^2}$ $\qquad$ $y=\pm\sqrt{1-(x-1)^2}$



$$\boxed{A=\int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx}$$

$x^2+y^2=2$

$x^2+0^2=2$

$x=\sqrt{2}$

use positive, b/c top of circle

(b) respect to $y \to$ solve for x.

$x^2+y^2=2$

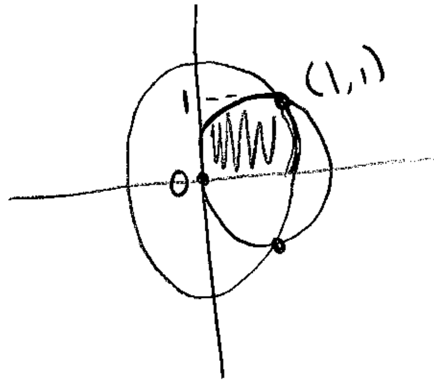
$x=\pm\sqrt{2-y^2}$

$(x-1)^2+y^2=1$

$(x-1)^2=1-y^2$

$x-1=\pm\sqrt{1-y^2}$

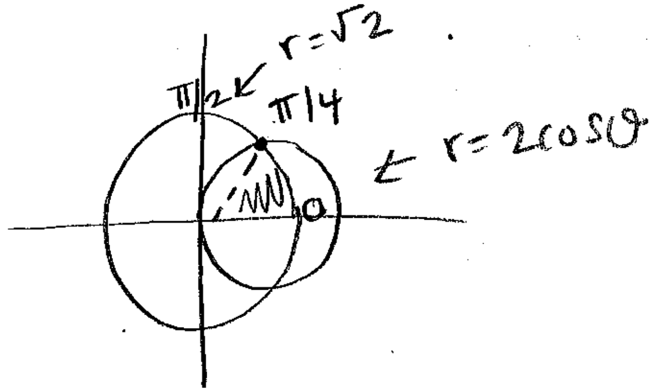
$x=1\pm\sqrt{1-y^2}$



$$\boxed{A=\int_0^1 \sqrt{2-y^2}-\left(1-\sqrt{1-y^2}\right)dy}$$

↑ use negative b/c left side of circle

(c) $r=\sqrt{2}$, $r=2\cos\theta$



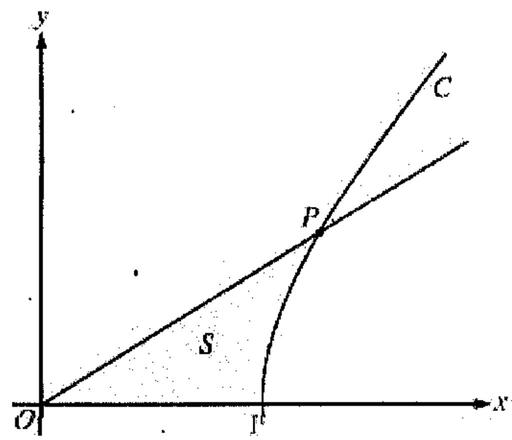
$\sqrt{2}=2\cos\theta$

$\frac{\sqrt{2}}{2}=\cos\theta$

$\theta=\pi/4$

$$\boxed{A=\frac{1}{2}\int_0^{\pi/4}(\sqrt{2})^2\,d\theta+\frac{1}{2}\int_{\pi/4}^{\pi/2}(2\cos\theta)^2\,d\theta}$$

2003 BC Exam - #3 – Calculator Active



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a)

$\frac{5}{3}y = \sqrt{1+y^2}$

$\frac{25}{9}y^2 = 1+y^2$

$\frac{16}{9}y^2 = 1$

$y^2 = \frac{9}{16}$

$y = 3/4$

$x = \frac{5}{3}y \rightarrow x = \frac{5}{3}\left(\frac{3}{4}\right)$

$x = \frac{15}{12} = \frac{5}{4}$

$\boxed{\left(\frac{5}{4}, \frac{3}{4}\right)}$

$\frac{dx}{dy}$ for curve C

$x = (1+y^2)^{1/2}$

$\frac{dx}{dy} = \cancel{\frac{1}{2}}(1+y^2)^{-1/2}(\cancel{2}y)$

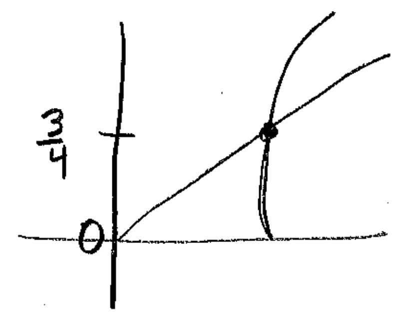
$= \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}$

at $\left(\frac{5}{4}, \frac{3}{4}\right) \rightarrow \frac{\frac{3}{4}}{\frac{5}{4}}$

$= \frac{3}{4}\cdot\frac{4}{5} = \boxed{\frac{3}{5}}$

(b) respect to y → solve for x

$x = \frac{5}{3}y \qquad x = \sqrt{1+y^2}$



$$\boxed{A = \int_0^{3/4} \sqrt{1+y^2} - \frac{5}{3}y \, dy = 0.347}$$

(c) $x^2 - y^2 = 1 \qquad x = r\cos\theta$ & $y = r\sin\theta$

$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2(\cos^2\theta - \sin^2\theta) = 1$

$$\boxed{r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}}$$

(d) $(\frac{5}{4}, \frac{3}{4})$ intersection, need θ

x y $\qquad \tan\theta = \frac{y}{x} \rightarrow \tan\theta = \frac{\frac{3}{4}}{\frac{5}{4}}$

$\theta = 0.540$

$$\boxed{A = \frac{1}{2}\int_0^{0.540} \frac{1}{\cos^2\theta - \sin^2\theta} \, d\theta}$$