AP Calculus BC
Unit 11 – Parametric Equations & Polar Coordinates

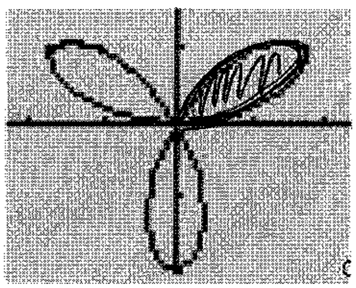
Day 7 Notes: Polar Graphs & Area (Day 1)

Area of a Sector:

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

Example 1:
Find the area of the region enclosed by one petal of the rose curve r = 2sin(3θ).

a. Graph the curve:



$0 \le \theta < \pi$

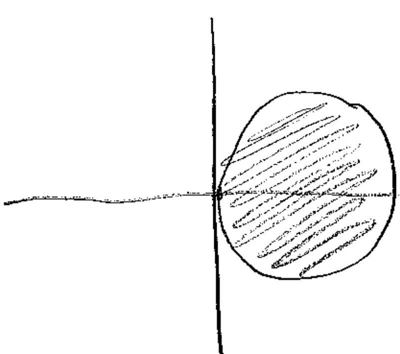
b. Locate an interval that traces out one petal.

one petal → $0 \le \theta < \pi/3$

c. Find the area:

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2 \, d\theta = \boxed{1.047}$$

Example 2:
Find the area of the region bounded by the graph of the polar equation r = 3cosθ. circle

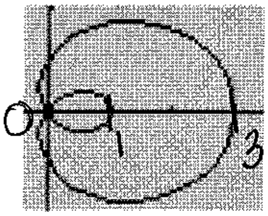


$0 \le \theta < \pi$

$$A = \frac{1}{2}\int_0^{\pi} (3\cos\theta)^2 \, d\theta = \boxed{7.069}$$

Example 3:
Find the area of the region between the loops of the limaçon $r = 1 + 2\cos\theta$.

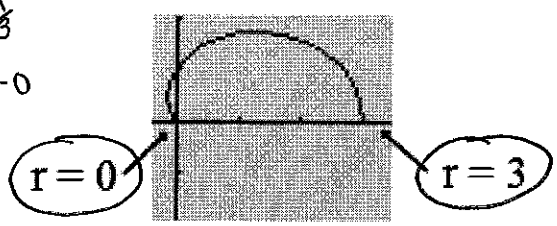
a. Graph:



b. Strategy: Area of outer loop – area of inner loop.

(There is symmetry to the graph, so let's find the area of half of each loop. We can double our answer at the end.)

c. Outer loop (half): Find limits, then find area.



$0 = 1 + 2\cos\theta$

$-\frac{1}{2} = \cos\theta$

$\theta = \frac{2\pi}{3}, \cancel{\frac{4\pi}{3}}$

$3 = 1 + 2\cos\theta$

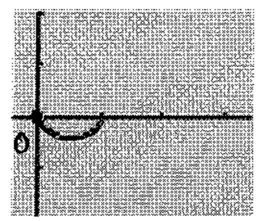
$1 = \cos\theta$

$\theta = 0$

$A = \frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2 \, d\theta = 4.441$

$4.441(2) = \boxed{8.882}$

d. Inner loop (half): Find limits, then find area.



$0 = 1 + 2\cos\theta$

$\frac{-1}{2} = \cos\theta$

halfway b/t $\rightarrow$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$A = \frac{1}{2}\int_{2\pi/3}^{\frac{3\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta = .272$

$.272(2) = \boxed{.544}$

e. Total area = ???? outer − inner

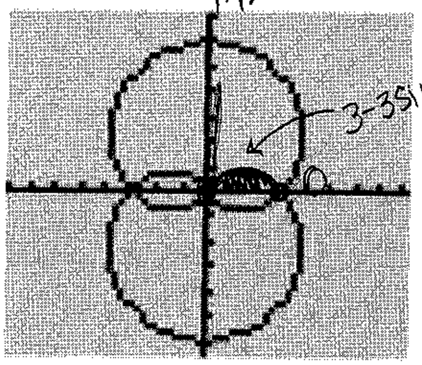
$8.882 - 0.544 = \boxed{8.338}$

Example 4:

Here's a tougher problem that requires finding points of intersection: Find the area of the common interior of the graphs of r = 3(1 + sinΘ) and r = 3(1 – sinΘ).

$r = 3 + 3\sin\theta \qquad r = 3 - 3\sin\theta$

a. Graph:



b. Find the points of intersection:

$\cancel{3} + 3\sin\theta = \cancel{3} - 3\sin\theta$

$3\sin\theta = -3\sin\theta$

$6\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0, \pi \rightarrow$

$r = 3 + 3\sin(0)$ $\boxed{(3, 0)}$

$r = 3 + 3\sin(\pi)$ $\boxed{(3, \pi)}$

@ pole $(0, \pi/2)$

$r = 0 \rightarrow 0 = 3 - 3\sin\theta$

$+1 = \sin\theta \quad \theta = \pi/2$ $\boxed{(0, \pi/2)}$

c. We really only need to find ¼ of the area! We can multiply by 4 at the end.

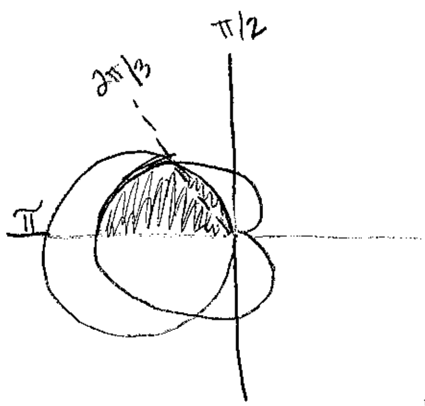
$$A = \frac{1}{2}\int_0^{\pi/2} (3 - 3\sin\theta)^2 \, d\theta = 1.603$$

$$1.603(4) = \boxed{6.412}$$

Example 5:

Find the area of the region common to the two regions bounded by the following curves.

r = -6cosθ and r = 2 – 2cosθ



Points of intersection:

$-6\cos\theta = 2 - 2\cos\theta$

$-4\cos\theta = 2$

$\cos\theta = -1/2$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$r = -6\cos(\frac{2\pi}{3})$ $\quad r = -6\cos(\frac{4\pi}{3})$

$r = 3$ $\quad r = 3$

$(3, 2\pi/3)$ $\quad (3, 4\pi/3)$

pole:

$r = 0$

$0 = -6\cos\theta$

$0 = \cos\theta$

$\theta = \pi/2$

$(0, \pi/2)$

circle on top

$$A = \frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 \, d\theta = 0.815$$

$$A = \frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2 \, d\theta = 7.039$$

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$.815 + 7.039 = 7.854$

$7.854(2) = \boxed{15.708}$