AP Calculus BC
Unit 11 – Parametric Equations & Polar Coordinates

# Day 5 Notes: Polar Graphs & Tangents

**Derivatives of Polar Equations**

A polar equation must first be converted into parametric form before the dy/dx can be found.

Remember the conversions $x = r\cos\theta$ and $y = r\sin\theta$? We can use these to find dy/dx.

$$\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}}$$

Note: This is the same as a parametric derivative!

**Example #1: Find dy/dx for r = 3 – 2cosθ when θ = 0.**

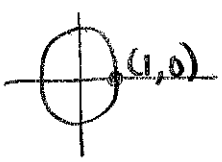
$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$x = (3-2\cos\theta)(\cos\theta)$ $\qquad$ $y = (3-2\cos\theta)(\sin\theta)$

$x = 3\cos\theta - 2\cos^2\theta$ $\qquad$ $y = 3\sin\theta - 2\cos\theta\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta + 2\sin\theta\sin\theta - 2\cos\theta\cos\theta}{-3\sin\theta - 4\cos\theta(-\sin\theta)} = \frac{3\cos\theta + 2\sin^2\theta - 2\cos^2\theta}{-3\sin\theta + 4\cos\theta\sin\theta}$$

at $\theta = 0$: $\dfrac{dy}{dx} = \dfrac{3(1) + \cancel{2(0)^2} - 2(1)^2}{-3\cancel{(0)} + 4(\cancel{1})(0)} = \dfrac{1}{0}$




$\boxed{\text{undefined}}$

**_Tangents_**

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

1. Horizontal tangents occur where $\frac{dy}{d\theta} = 0$.
2. Vertical tangents occur where $\frac{dx}{d\theta} = 0$.
3. If $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$ simultaneously...... cusp?

$(r, \theta)$

**Example #2**: **Find the points (~~ ~~) of horizontal and vertical tangency for the polar curve $r = 4\cos\theta$.**

$x = (4\cos\theta)(\cos\theta)$ $\quad$ $y = (4\cos\theta)(\sin\theta)$

$x = 4\cos^2\theta$ $\quad$ $y = \underbrace{4\cos\theta}\underbrace{\sin\theta}$

horizontal: $\frac{dy}{d\theta} = \theta \rightarrow -4\sin\theta\sin\theta + 4\cos\theta\cos\theta = 0$

$-4\sin^2\theta + 4\cos^2\theta = 0$

$4(-\sin^2\theta + \cos^2\theta) = 0$

$\cos^2\theta - \sin^2\theta = 0$

$\sin^2\theta + \cos^2\theta = 1$

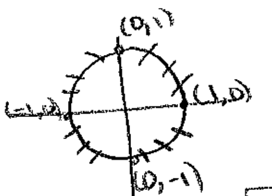
$\cos^2\theta - (1 - \cos^2\theta) = 0$

$\cos^2\theta - 1 + \cos^2\theta = 0$

$2\cos^2\theta - 1 = 0$

$\cos^2\theta = 1/2$

$\cos\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$




$r = 4\cos(\frac{\pi}{4}) = 4(\frac{\sqrt{2}}{2})$ $\quad \boxed{(2\sqrt{2}, \pi/4)}$

$r = 4\cos(\frac{3\pi}{4}) = 4(-\frac{\sqrt{2}}{2})$ $\quad \boxed{(-2\sqrt{2}, 3\pi/4)}$

$r = 4\cos(5\pi/4)$ $\quad \boxed{(-2\sqrt{2}, 5\pi/4)}$

$r = 4\cos(7\pi/4)$ $\quad \boxed{(2\sqrt{2}, 7\pi/4)}$

$\theta = \pi/4, 3\pi/4, \frac{5\pi}{4}, \frac{7\pi}{4}$

vertical:

$\frac{dx}{d\theta} = 0$ $\quad 8\cos\theta\sin\theta = 0$

$\cos\theta\sin\theta = 0$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

$r = 4\cos\theta$

$r = 4\cos 0 = 4$ $\quad \boxed{(4, 0)}$

$r = 4\cos(\frac{\pi}{2}) = 0$ $\quad \boxed{(0, \pi/2)}$

$r = 4\cos(\pi) = -4$ $\quad \boxed{(-4, \pi)}$

$r = 4\cos(\frac{3\pi}{2}) = 0$ $\quad \boxed{(0, 3\pi/2)}$

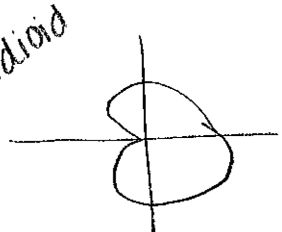
***Tangents at the pole***

1. At the pole, r = 0. Find the values of θ where r = 0.
2. The radial lines θ = α will be the tangents at the pole.

* It's possible to have more than one tangent at the pole.

**Example #3**: **Find the tangents at the pole: $r = 3 + 3\cos\theta$.**

cardioid



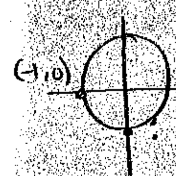
$0 \le \theta < 2\pi$

$3 + 3\cos\theta = 0$

$3\cos\theta = -3$

$\cos\theta = -1$

$\boxed{\theta = \pi}$

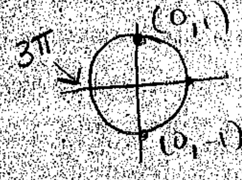



**Example #4**: **Find the tangents at the pole for the curve $r = 2\cos(3\theta)$.**

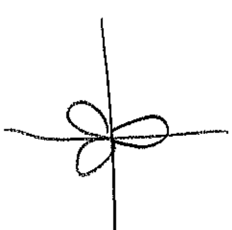
$2\cos(3\theta) = 0$

$\cos(3\theta) = 0$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$




rose curve



$0 \le \theta < \pi$

$0 \le 3\theta < 3\pi$

$3\theta = \frac{\pi}{2}$ $\quad$ $3\theta = \frac{3\pi}{2}$ $\quad$ $3\theta = \frac{5\pi}{2}$

$\boxed{\theta = \frac{\pi}{6}}$ $\quad$ $\theta = \frac{3\pi}{6}$ $\quad$ $\boxed{\theta = \frac{5\pi}{6}}$

$\boxed{\theta = \pi/2}$