Day 2 Notes: Parametric Equations & Calculus

DERIVATIVES OF PARAMETRIC EQUATIONS

If $x = f(t)$ and $y = g(t)$ represent curve C, then the slope of C at point (x, y) is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ where } \frac{dx}{dt} \neq 0.$$

Example: Find the equation of the tangent to the curve defined by $x = \sqrt{t}$ and $y = \sqrt{t-1}$ when $t = 2$.

$x = t^{1/2} \qquad y = (t-1)^{1/2}$

$\frac{dy}{dx} = \frac{\cancel{\frac{1}{2}}(t-1)^{-1/2}}{\cancel{\frac{1}{2}}t^{-1/2}} = \frac{\sqrt{t}}{\sqrt{t-1}} \rightarrow$ slope: $\frac{\sqrt{2}}{\sqrt{2-1}} = \sqrt{2}$

point: $x = \sqrt{2}, y = \sqrt{2-1} = 1 \quad (\sqrt{2}, 1)$

$\boxed{y - 1 = \sqrt{2}(x - \sqrt{2})}$

HIGHER ORDER DERIVATIVES

1$^{\text{ST}}$ derivative: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

2$^{\text{nd}}$ derivative: $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\boxed{\frac{dy}{dx}}\right]}{\boxed{dx/dt}}$

deriv. of 1st

same as 1st deriv.

Example: Find $\frac{d^2y}{dx^2}$ if $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$.

$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$

$\frac{d^2y}{dx^2} = \frac{\frac{(1-\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}}{1-\cos\theta}$

$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^3} = \frac{\cos\theta - 1(\cos^2\theta + \sin^2\theta)}{(1-\cos\theta)^3}$

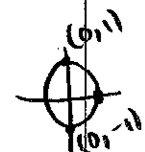
$= \frac{-1(1-\cos\theta)}{(1-\cos\theta)^3} = \boxed{\frac{-1}{(1-\cos\theta)^2}}$

HORIZONTAL & VERTICAL TANGENTS

- **Horizontal:** Since $\frac{dy}{dx} = 0$ implies a horizontal tangent, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, horizontal tangents will occur when $\frac{dy}{dt} = 0$
- **Vertical:** Vertical tangents occur where $\frac{dy}{dx}$ is undefined. This happens when $\boxed{\frac{dx}{dt} = 0}$
- **Neither:** If $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are simultaneously equal to 0 for the same value of t, we will usually have a sharp turn or a cusp, NOT a horizontal or vertical tangent.

Example: Find all points of horizontal and vertical tangency to the curve given by $x = \cos\theta$, $y = 2\sin\theta$

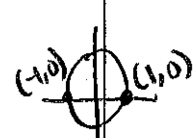
horizontal $\quad \frac{dy}{dt} = 0 \rightarrow 2\cos\theta = 0$
$\cos\theta = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$



$\theta = \frac{\pi}{2} \rightarrow x = \cos(\frac{\pi}{2}) = 0$
$y = 2\sin(\frac{\pi}{2}) = 2 \quad \boxed{(0,2)}$

$\theta = \frac{3\pi}{2} \rightarrow x = \cos(\frac{3\pi}{2}) = 0 \quad \boxed{(0,-2)}$
$y = 2\sin(\frac{3\pi}{2}) = -2$

vertical $\quad \frac{dx}{dt} = 0 \rightarrow -\sin\theta = 0$
$\sin\theta = 0$
$\theta = 0, \pi$



$\theta = 0 \rightarrow x = \cos(0) = 1$
$y = 2\sin(0) = 0 \quad \boxed{(1,0)}$

$\theta = \pi \rightarrow x = \cos(\pi) = -1$
$y = 2\sin(\pi) = 0 \quad \boxed{(-1,0)}$

Another example: The prolate cycloid given by $x = 2t - \pi \sin t$ and $y = 2 - \pi \cos t$ crosses over itself at the point $(0, 2)$. (Graph to confirm! Set T-min and T-max so that $-2\pi \le t \le 2\pi$) Find the equations of both tangent lines at this point.

$$\frac{dy}{dx} = \frac{\pi \sin t}{2 - \pi \cos t}$$

Need to find t:

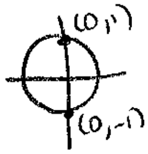
$(0, 2)$

$2 = 2 - \pi \cos t$

$0 = -\pi \cos t$

$0 = \cos t$

$t = \pi/2,\ 3\pi/2$



$t = \pi/2 \rightarrow$ Slope $= \dfrac{\pi \sin(\frac{\pi}{2})}{2 - \pi\cos(\frac{\pi}{2})} = \dfrac{\pi(1)}{2 - \pi(0)} = \dfrac{\pi}{2}$

$$\boxed{y - 2 = \frac{\pi}{2}(x - 0)}$$

$t = 3\pi/2 \rightarrow$ slope $= \dfrac{\pi \sin(\frac{3\pi}{2})}{2 - \pi\cos(\frac{3\pi}{2})} = \dfrac{\pi(-1)}{2 - \pi(0)} = \dfrac{-\pi}{2}$

$$\boxed{y - 2 = -\frac{\pi}{2}(x - 0)}$$

ARC LENGTH OF PARAMETRICALLY DEFINED CURVES

$$s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

*Arc length gives the **total distance traveled.**

Distance should be (+)

$u = 1 + 36t^2$

$du = 72t\,dt$

Example: Find the $\boxed{\text{total distance}}$ a particle travels along a path given by $x = t^2 + 1$ and $y = 4t^3 + 3$ on the interval $-1 \le t \le 0$.

$\int_{-1}^{0} \sqrt{(2t)^2 + (12t^2)^2}\,dt$

$\int_{-1}^{0} \sqrt{4t^2 + 144t^4}\,dt$

$\sqrt{4t^2(1 + 36t^2)}$

$\frac{1}{36}\int_{-1}^{0} 36 \cdot 2t\sqrt{1 + 36t^2}\,dt$

$\frac{1}{36}\int_{37}^{1} u^{1/2}\,du$

$-\frac{1}{36}\int_{1}^{37} u^{1/2}\,du$

$-\frac{1}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{37}$

$\frac{-2}{108}\left(37^{3/2} - 1\right)$

$\boxed{+\frac{1}{54}\left(37^{3/2} - 1\right)} = \boxed{4.149}$

FINDING THE POSITION OF A PARTICLE

Position of the particle uses the ***Fundamental Theorem of Calculus!***

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(t)\ dt$$

or

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} y'(t)\ dt$$

Example: The position of a particle in the xy-plane is given by $(x(t), y(t))$, with $\dfrac{dx}{dt} = t^2 + \sin(3t^2)$. At $t = 0$, the particle at the point (5, 1). Find the x-coordinate of the particle at $t = 3$.

$x(0) = 5$

$x(3) = ?$

$x(3) - x(0) = \underbrace{\int_0^3 t^2 + \sin(3t^2)\,dt}_{\text{calc.}}$

$x(3) - 5 = 9.377$

$$\boxed{x(3) = 14.377}$$

$X(t) = 2(t-3)^{1/2}$

SPEED OF A PARTICLE

Speed $= \sqrt{[x'(t)]^2 + [y'(t)]^2}$

Example: A particle follows a path defined parametrically by $x(t) = 2\sqrt{t-3}$, $y(t) = 3t^2$. What is the speed of the particle at t = 7?

$$\text{Speed} = \sqrt{[(t-3)^{-1/2}]^2 + [6t]^2}$$

$$\sqrt{(t-3)^{-1} + 36t^2}$$

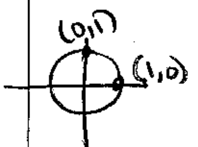
$$t = 7 \to \sqrt{(7-3)^{-1} + 36(7)^2}$$

$$= \boxed{42.003}$$

AREA OF PARAMETRICALLY DEFINED REGIONS

$$A = \int_a^b y\, dx \quad \text{deriv. of } x$$

***Note: Since we are integrating with respect to x, the limits a and b are x-values. Use these to find the corresponding values of θ or t.



$X = \sin\theta$

$0 = \sin\theta$

$\theta = 0$

$\sin(0) = 0$

$\int_0^{\pi/2}$

$1 = \sin\theta$

$\theta = \pi/2$

Example: Find the area of the region enclosed by the graph of $\boxed{x = \sin\theta}$, $y = \sin^2\theta$, the x-axis and the vertical line $x = 1$.

$X = 0$

$$\int_0^1 y\, dx$$

$$A = \int_0^{\pi/2} \sin^2\theta(\cos\theta)\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\int u^2\, du$$

$$\frac{1}{3}u^3 = \frac{1}{3}\sin^3\theta \Big]_0^{\pi/2}$$

$$\frac{1}{3}\sin^3\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin^3(0)$$

$$\frac{1}{3}(1) - \frac{1}{3}(0) = \boxed{\frac{1}{3}}$$

VOLUME OF PARAMETRICALLY DEFINED REGIONS

$$V = \pi\int_a^b y^2\, dx$$

Example: Suppose the region described in the previous example is rotated about the x-axis. Find the volume of the resulting solid.

$$V = \pi\int_0^{\pi/2} \sin^4\theta \cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\pi\int u^4\, du$$

$$\frac{\pi}{5}u^5 = \frac{\pi}{5}\sin^5\theta \Big]_0^{\pi/2}$$

$$\frac{\pi}{5}\sin^5\left(\frac{\pi}{2}\right) - \frac{\pi}{5}\sin^5(0)$$

$$\frac{\pi}{5}(1) - \frac{\pi}{5}(0) = \boxed{\frac{\pi}{5}}$$