AP Calculus BC
Unit 11 – Parametric Equations & Polar Coordinates

# Day 1 Notes: Parametric Equations

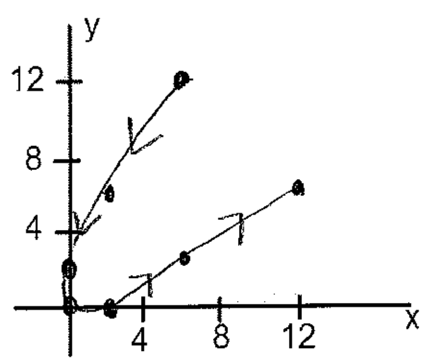
**Parametric Equations introduce a "parameter", frequently $t$ (time) or $\theta$ (angle).

** We can tell where an object will be and at what time it will be there.

**Parametric equations give position, speed, and direction.

**Example 1**: Use the table below to help you graph the parametric equations $x(t) = t^2 + t$ and $y(t) = t^2 - t$.

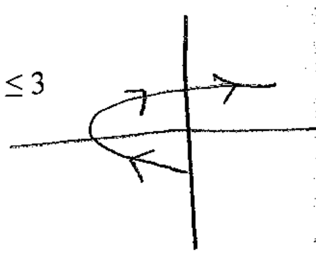
| t | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| x | 6 | 2 | 0 | 0 | 2 | 6 | 12 |
| y | 12 | 6 | 2 | 0 | 0 | 2 | 6 |




Note that when (x, y) is plotted according to ascending values of $t$, the curve is traced out in a specific direction called the **orientation**.

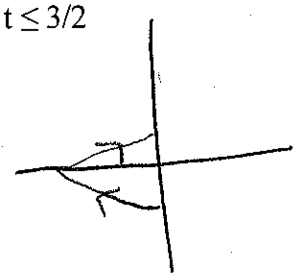
**Example 2**: Graphing Parametric Equations in your calculator.
*Set your calculator to parametric mode.

a) Graph $x = t^2 - 4$, $y = t/2$, for $-2 \le t \le 3$



* watch how your calc graphs to get the direction

b) Graph $x = 4t^2 - 4$, $y = t$, for $-1 \le t \le 3/2$



x & y

**There are times when we might want to convert parametric equations to rectangular, and vice versa.**

1) **Solve one of the equations for t.**
2) **Substitute t into the other equation.**
3) **Adjust the domain of the rectangular equation to fit that of the parametric equations.**

**Example 3**: Convert x = t – 1 and y = t / (t – 1) to rectangular form.

$$x+1=t \longrightarrow y=\frac{x+1}{x+1-1} \rightarrow \boxed{y=\frac{x+1}{x}}$$

**Example 4**: Convert x = 3t – 1 and y = 2t – 1 to rectangular form.

$$\frac{x+1}{3}=t \longrightarrow y=2\left(\frac{x+1}{3}\right)-1$$

$$y=\frac{2}{3}(x+1)-1 \quad y=\frac{2}{3}x+\frac{2}{3}-1 \quad \boxed{y=\frac{2}{3}x-\frac{1}{3}}$$

***Remember: $\cos^2\theta + \sin^2\theta = 1$ and $\sec^2\theta - \tan^2\theta = 1$*

**Example 5**: Convert to rectangular form: x = 5cosθ, y = 5sinθ circle

$$\frac{x}{5}=\cos\theta \qquad \frac{y}{5}=\sin\theta$$

$$\cos^2\theta+\sin^2\theta=1$$

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{5}\right)^2=1 \rightarrow \frac{x^2}{25}+\frac{y^2}{25}=1 \rightarrow \boxed{x^2+y^2=25}$$

ellipse

**Example 6**: Convert to rectangular form: x = 4sin(2θ), y = 2cos(2θ)

$$\frac{x}{4}=\sin 2\theta \qquad \frac{y}{2}=\cos 2\theta$$

$$\cos^2(2\theta)+\sin^2(2\theta)=1$$

$$\left(\frac{y}{2}\right)^2+\left(\frac{x}{4}\right)^2=1 \rightarrow \boxed{\frac{y^2}{4}+\frac{x^2}{16}=1}$$

ellipse

**Example 7**: Convert to parametric form: $\frac{(x-4)^2}{25}+\frac{(y+1)^2}{9}=1$

$$\frac{(x-4)^2}{25}=\cos^2\theta \qquad \frac{(y+1)^2}{9}=\sin^2\theta$$

$$\frac{x-4}{5}=\cos\theta \qquad \frac{y+1}{3}=\sin\theta$$

$$x-4=5\cos\theta \quad \boxed{x=4+5\cos\theta} \qquad \boxed{y=-1+3\sin\theta}$$

hyperbola

**Example 8**: Convert to parametric form: $\frac{y^2}{14}-\frac{x^2}{9}=1$

$$\sec^2\theta-\tan^2\theta=1$$

$$\frac{y^2}{14}=\sec^2\theta \qquad \frac{x^2}{9}=\tan^2\theta$$

$$\frac{y}{\sqrt{14}}=\sec\theta \qquad \frac{x}{3}=\tan\theta$$

$$\boxed{y=\sqrt{14}\sec\theta} \qquad \boxed{x=3\tan\theta}$$

Not smooth = cusps / sharp turns

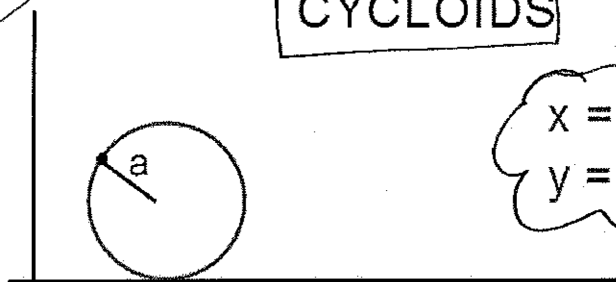
Derivatives of x and y are 0 = not smooth

Definition of <u>SMOOTH CURVE</u>

If C is a curve represented by $x = f(t)$, $y = g(t)$ on an interval [a, b], and $f'$ and $g'$ are continuous on [a, b] and not simultaneously equal to 0 (except maybe at the endpoints of [a, b]), then C is a smooth curve.

*A curve is not smooth wherever it has cusps or sharp turns.

## CYCLOIDS




$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

Cycloids are formed if we look at the motion of a point on a circle with radius a as it rolls along a line.

**Cycloids have sharp turns at $x = 2n\pi a$.**

not smooth at $x = 2n\pi$

why?

$x = a(\theta - \sin\theta) = a\theta - a\sin\theta$

$x' = a - a\cos\theta$

$x'(2\pi n) = 0$

$y = a(1 - \cos\theta) = a - a\cos\theta$

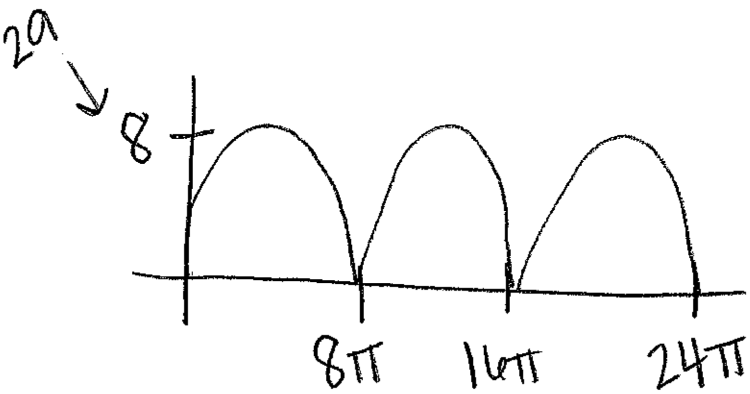
$y' = a\sin\theta$

$y'(2\pi n) = 0$

$a = 4$

**Example 9:** Sketch the cycloid $x = 4(\theta - \sin\theta)$, $y = 4(1 - \cos\theta)$. Identify any points at which the curve is not smooth.

sharp turns $= 2\pi n(4)$
$= 8\pi n$




Not smooth at $2n\pi$