AP CALCULUS BC NAME Answer Key*
Unit 10 - Review

#'s 1 – 11: Multiple Choice & #'s 12 – 13: Free Response * means no calculator!

*1. The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which is a power series expansion for $\frac{x^2}{1-x^2}$?

A. $1+x^2+x^4+x^6+x^8+...$ B. $x^2+x^3+x^4+x^5+...$ C. $x^2+2x^3+3x^4+4x^5+...$
D. $x^2+x^4+x^6+x^8+...$ E. $x^2-x^4+x^6-x^8+...$

*2. A function f has Maclaurin series given by $\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}+...+\frac{x^{n+3}}{(n+1)!}+...$. Which of the following is an expression for $f(x)$?

A. $-3x\sin x+3x^2$ B. $-\cos(x^2)+1$ C. $-x^2\cos x+x^2$
D. $x^2e^x-x^3-x^2$ E. $e^{x^2}-x^2-1$

*3. What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x=0$?

A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. 1 D. 3 E. 6

*4. What is the approximation of the value of sin 1 obtained by using the fifth-degree Taylor Polynomial about $x=0$ for $\sin x$?

A. $1-\frac{1}{2}+\frac{1}{24}$ B. $1-\frac{1}{2}+\frac{1}{4}$ C. $1-\frac{1}{3}+\frac{1}{5}$
D. $1-\frac{1}{4}+\frac{1}{8}$ E. $1-\frac{1}{6}+\frac{1}{120}$

*5. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1)=$

A. 0 B. a_1 C. $\sum_{n=0}^{\infty} a_n$ D. $\sum_{n=1}^{\infty} na_n$ E. $\sum_{n=1}^{\infty} na_n^{\,n-1}$

6. Let $P(x)=3x^2-5x^3+7x^4+3x^5$ be the fifth-degree Taylor polynomial for the function f about $x=0$. What is the value of $f'''(0)$?

A. -30 B. -15 C. -5 D. $-\frac{5}{6}$ E. $-\frac{1}{6}$

*7. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ converges?

A. $-\frac{5}{2}<x<-\frac{1}{2}$ B. $-\frac{5}{2}<x\le-\frac{1}{2}$ C. $-\frac{5}{2}\le x<-\frac{1}{2}$
D. $-\frac{1}{2}<x<-\frac{1}{2}$ E. $x\le-\frac{1}{2}$

*8. Which of the following is the Maclaurin series for e^{3x}?

A. $1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

B. $3+9x+\frac{27x^2}{2}+\frac{81x^3}{3!}+\frac{243x^4}{4!}+\cdots$

C. $1-3x+\frac{9x^2}{2}-\frac{27x^3}{3!}+\frac{81x^4}{4!}+\cdots$

D. $1+3x+\frac{3x^2}{2}+\frac{3x^3}{3!}+\frac{3x^4}{4!}+\cdots$

E. $1+3x+\frac{9x^2}{2}+\frac{27x^3}{3!}+\frac{81x^4}{4!}+\cdots$

*9. What is the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 2^n}$?

A. $1<x<5$ B. $1\le x<5$ C. $1\le x\le 5$ D. $2<x<4$ E. $2\le x\le 4$

*10. The third-degree Taylor polynomial for a function f about x = 4 is $\frac{(x-4)^3}{512}-\frac{(x-4)^2}{64}+\frac{(x-4)}{4}+2$. What is the value of $f'''(4)$?

A. $-\frac{1}{64}$ B. $-\frac{1}{32}$ C. $\frac{1}{512}$ D. $\frac{3}{256}$ E. $\frac{81}{256}$

*11. Which of the following is the Maclaurin series for $\frac{1}{(1-x)^2}$?

A. $1-x+x^2-x^3+\cdots$ B. $1-2x+3x^2-4x^3+\cdots$ C. $1+2x+3x^2+4x^3+\cdots$

D. $1+x^2+x^4+x^6+\cdots$ E. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$

*12. Let f be the function given by $f(x)=\sin\left(5x+\frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$.

a. Find $P(x)$.

b. Find the coefficient of x^{22} in the Taylor series for f about $x=0$.

c. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{100}$.

d. Let G be the function given by $G(x)=\int_0^x f(t)dt$. Write the third-degree Taylor polynomial for G about $x=0$.

*13. The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\ldots$$ for all real numbers x.

a. Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x=0$.

b. Show that $1-\frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

c. Show that $y=f(x)$ is a solution to the differential equation $xy'+y=\cos x$

① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

$$\frac{x^2}{1-x^2} = x^2 \sum_{n=0}^{\infty} x^{2n} = x^2\left[1 + x^2 + x^4 + x^6 + \dots\right]$$

$$= \boxed{x^2 + x^4 + x^6 + x^8 + \dots \text{ D.}}$$

② $\frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots + \frac{x^{n+3}}{(n+1)!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^2 e^x = x^2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right]$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

$$x^2 e^x - x^3 - x^2 = \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

$$\boxed{\text{D. } x^2 e^x - x^3 - x^2}$$

(3) $f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2} \rightarrow \frac{1}{(1+x)^2}$

$f'(x) = -2(1+x)^{-3} \rightarrow \frac{-2}{(1+x)^3}$

$f''(x) = 6(1+x)^{-4} \rightarrow \frac{6}{(1+x)^4} \rightarrow f''(0) = \frac{6}{(1+0)^4} = 6$

$\frac{6x^2}{2!} = 3x^2$ $\boxed{D. 3}$

(4) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\frac{\cancel{(-1)^0}^{1} x^{2(0)+1}}{(2(0)+1)!} + \frac{(-1)^1 x^{2(1)+1}}{(2(1)+1)!} + \frac{(-1)^2 x^{2(2)+1}}{(2(2)+1)!}$

$x + \frac{-x^3}{3!} + \frac{x^5}{5!}$

$x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$\sin(1) = 1 - \frac{1}{6}(1)^3 + \frac{1}{120}(1)^5$

$= \boxed{1 - \frac{1}{6} + \frac{1}{120} = E.}$

(5) $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$f'(1) = \sum_{n=0}^{\infty} n a_n (1)^{n-1}$

$= \boxed{\sum_{n=1}^{\infty} n a_n = D.}$

(6) $P(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5 = f(x)$

$f'(x) = 6x - 15x^2 + 28x^3 + 15x^4$

$f''(x) = 6 - 30x + 84x^2 + 60x^3$

$f'''(x) = 0 - 30 + 168x + 180x^2$

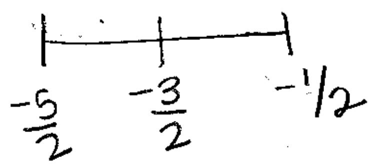
$\boxed{f'''(0) = -30 = A}$

(7) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ center $= -\frac{3}{2}$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\left(x+\frac{3}{2}\right)^{n+1}}{(n+1)}\cdot\frac{n}{\cancel{(-1)^n}\cancel{\left(x+\frac{3}{2}\right)^n}}\right| = \lim_{n\to\infty}\left|\frac{\left(x+\frac{3}{2}\right)n}{(n+1)}\right|$$

$$\left|x+\frac{3}{2}\right|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \left|x+\frac{3}{2}\right|$$

$$\left|x+\frac{3}{2}\right| < 1 \quad R=1$$



$\underline{x=-5/2}$: $\sum_1 \frac{(-1)^n(-1)^n}{n} = \sum_1 \frac{1}{n}$ p-series diverges

$\underline{x=-1/2}$: $\sum_1 \frac{(-1)^n(1)^n}{n} = \sum_1 \frac{(-1)^n}{n}$ alt. series:

$\lim_{n\to\infty}\frac{1}{n} = 0$ ✓

$\frac{1}{n+1} \leq \frac{1}{n}$ ✓

conv.

$$\boxed{\left(-\frac{5}{2}, -\frac{1}{2}\right] = B}$$

⑧ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$

$\underline{n=0}: \frac{(3x)^0}{0!} = 1$

$\underline{n=1}: \frac{(3x)^1}{1!} = 3x$

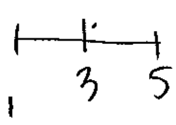
$\underline{n=2}: \frac{(3x)^2}{2!} = \frac{9x^2}{2}$ E.

$\underline{n=3}\ \frac{(3x)^3}{3!} = \frac{27x^3}{3!}$

⑨ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$ $\lim_{n\to\infty} \left| \frac{(x-3)^{\cancel{n+1}}}{(n+1)2^{\cancel{n+1}}} \cdot \frac{(n)(\cancel{2^n})}{\cancel{(x-3)^n}} \right|$

$\lim_{n\to\infty} \left| \frac{(x-3)(n)}{(n+1)(2)} \right| = \left|\frac{x-3}{2}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \left|\frac{x-3}{2}\right| < 1$

$R = 2$



$\underline{x=1}: \sum \frac{(-2)^n}{n2^n} = \sum \frac{(-1)^n}{n}$ alt series test

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ converges

$\frac{1}{n+1} \le \frac{1}{n}$ ✓

$\underline{x=5}: \sum \frac{(2)^n}{n2^n} = \sum \frac{1}{n}$ p-series test div.

$\boxed{[1, 5) = B}$

(10) $f(x) = \frac{(x-4)^3}{512} - \frac{(x-4)^2}{64} + \frac{(x-4)}{4} + 2$

$f'(x) = \frac{3}{512}(x-4)^2 - \frac{2}{64}(x-4) + \frac{1}{4} + 0$

$f''(x) = \frac{6}{512}(x-4) - \frac{2}{64} + 0 + 0$

$f'''(x) = \frac{6}{512}$

$f'''(4) = \frac{6}{512} = \boxed{\frac{3}{256} = 0}$

(11) $f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2} = \frac{1}{(1-x)^2} = \frac{1}{(1-0)^2} = 1$

$f''(x) = -2(1-x)^{-3} = \frac{-2}{(1-x)^3} = \frac{-2}{(1-0)^3} = -2$

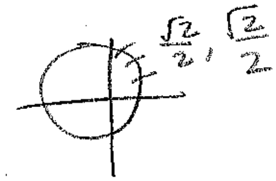
$f'''(x) = 6(1-x)^{-4} = \frac{6}{(1-x)^4} = \frac{6}{(1-0)^4} = 6$

$f^4(x) = -24(1-x)^{-5} = \frac{-24}{(1-x)^5} = \frac{-24}{(1-0)^5} = -24$

$1 + -2x + \frac{6x^2}{2!} - \frac{24x^3}{3!} + \ldots$

$\boxed{1 - 2x + 3x^2 - 4x^3 + \ldots \quad B}$

(12) $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$



ⓐ $P(x) \to$ 3rd degree Taylor w/ center $x = 0$

$f(x) = \sin\left(5x + \frac{\pi}{4}\right) \to f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = 5\cos\left(5x + \frac{\pi}{4}\right) \to f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$

$f''(x) = -25\sin\left(5x + \frac{\pi}{4}\right) \to f''(0) = -25\sin\left(\frac{\pi}{4}\right) = \frac{-25\sqrt{2}}{2}$

$f'''(x) = -125\cos\left(5x + \frac{\pi}{4}\right) \to f'''(0) = -125\sin\left(\frac{\pi}{4}\right) = \frac{-125\sqrt{2}}{2}$

$$\boxed{P(x) = \underset{(0)}{\frac{\sqrt{2}}{2}} + \underset{(1)}{\frac{5\sqrt{2}}{2}x} + \underset{(2)}{\frac{-25\sqrt{2}}{2} \cdot \frac{x^2}{2!}} \underset{(3)}{- \frac{125\sqrt{2}}{2 \cdot 3!}x^3}}$$

+ 4 5
− 6 7
+ 8 9
− 10 11
+ 12 13
− 14 15
+ 16 17
− 18 19
+ 20 21
− 22

ⓑ $$\boxed{\frac{-5^{22}\sqrt{2}}{2(22!)}}$$

$f^4(x) = \overset{\text{max}}{625}\sin\left(5x + \frac{\pi}{4}\right)$

ⓒ $\frac{f^4(c)}{4!}\left(\frac{1}{10}\right)^4 = \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \boxed{\frac{1}{384} < \frac{1}{100}}$ ✓

← max from 0 to $\frac{1}{10}$

ⓓ $G(x) = \int_0^x f(t)\,dt = \int_0^x \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{2 \cdot 2!}t^2$

$\left.\frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{4}t^2 - \frac{25\sqrt{2}}{6 \cdot 2!}t^3\right]_0^x$

$$\boxed{\frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3}$$

(13) $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$

(a) $f(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6 + \ldots$

$f'(x) = 0 - \frac{3}{6}x + \frac{5}{120}x^3 - \frac{6}{5040}x^5 + \ldots$

$f''(x) = 0 - \frac{3}{6} + \frac{15}{120}x^2 - \frac{30}{5040}x^4 + \ldots$

$f'(0) = 0 \leftarrow$ critical pt at $x = 0$

$f''(x) = -\frac{1}{2}$

Since $f'(0) = 0$ and $f''(0) < 0$, $x = 0$ is a local maximum

(b) $f(1) = 1 - \frac{1^2}{3!} + \frac{1^4}{5!} - \frac{1^6}{7!} + \ldots$

This is an alternating series whose terms decrease in absolute value to 0.

$\Rightarrow$ The error is less than the "next term"

$\frac{1^4}{5!} = \frac{1}{120} < \frac{1}{100}$ ✓

(c) $y = f(x)$

$$xy' + y = \cos x$$

$$y' = f'(x) = 0 - \frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \ldots$$

$$x(y') = x\left(\frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \ldots\right)$$

$$= \frac{-2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \ldots$$

$$xy' + y = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \ldots + 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$$

$$= 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} - \frac{1}{7!}\right)x^6$$

$$= 1 - \left(\frac{3}{3!}\right)x^2 + \left(\frac{5}{5!}\right)x^4 - \frac{7}{7!}x^6$$

$$= \underset{0}{1} - \underset{1}{\frac{1}{2!}x^2} + \underset{2}{\frac{1}{4!}x^4} - \underset{3}{\frac{1}{6!}x^6} + \ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \boxed{\cos x} \checkmark$$