TAYLOR SERIES

TAYLOR POLYNOMIALS: Use an nth degree Taylor polynomial to approximate *f(x)* near *x=c*.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

*A Maclaurin polynomial is centered at $x = 0$.

***Remember**: Don't go beyond the degree asked for on the test!*

TAYLOR'S THEOREM:

$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$, where

$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$. ← (Lagrange error bound) You don't need to find *z*. Just look for the max value of $f^{(n+1)}(z)$ on the interval between *x* and *c*.

TAYLOR & MACLAURIN SERIES:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots.$$

Taylor series converge to a function *f(x)* if $R_n(x) \to 0$ as $n \to \infty$.

SAVE TIME ON THE EXAM BY MEMORIZING THESE COMMON TAYLOR SERIES!

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots. + \frac{(-1)^{n-1}(x-1)^n}{n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots. + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots. = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

CONVERGENCE: Use ratio test to determine convergence of Taylor series.

If a_n is the general term of a Taylor series,

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ the series converges
2. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1 \Rightarrow$ the series diverges

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INTEGRALS & DERIVATIVES OF TAYLOR SERIES can be calculated term by term from a known Taylor series. Integrals and derivatives have

- the same radius of convergence
- the same interval of convergence, except maybe at the endpoints of the interval. (You must test the endpoints in the series to determine convergence there.)

AP CALCULUS BC **NAME ________________________________**
Unit 10 - Review

#'s 1 – 11: Multiple Choice & #'s 12 – 13: Free Response * means no calculator!

*1. The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which is a power series expansion for $\frac{x^2}{1-x^2}$?

A. $1+x^2+x^4+x^6+x^8+\ldots$ B. $x^2+x^3+x^4+x^5+\ldots$ C. $x^2+2x^3+3x^4+4x^5+\ldots$
D. $x^2+x^4+x^6+x^8+\ldots$ E. $x^2-x^4+x^6-x^8+\ldots$

*2. A function f has Maclaurin series given by $\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}+\ldots+\frac{x^{n+3}}{(n+1)!}+\ldots$. Which of the following is an expression for *f(x)*?

A. $-3x\sin x+3x^2$ B. $-\cos(x^2)+1$ C. $-x^2\cos x+x^2$
D. $x^2e^x-x^3-x^2$ E. $e^{x^2}-x^2-1$

*3. What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x = 0$?

A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. 1 D. 3 E. 6

*4. What is the approximation of the value of sin 1 obtained by using the fifth-degree Taylor Polynomial about $x = 0$ for sin x?

A. $1-\frac{1}{2}+\frac{1}{24}$ B. $1-\frac{1}{2}+\frac{1}{4}$ C. $1-\frac{1}{3}+\frac{1}{5}$
D. $1-\frac{1}{4}+\frac{1}{8}$ E. $1-\frac{1}{6}+\frac{1}{120}$

*5. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to *f(x)* for all real *x*, then $f'(1) =$

A. 0 B. a_1 C. $\sum_{n=0}^{\infty} a_n$ D. $\sum_{n=1}^{\infty} na_n$ E. $\sum_{n=1}^{\infty} na_n{}^{n-1}$

6. Let $P(x)=3x^2-5x^3+7x^4+3x^5$ be the fifth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f'''(0)$?

A. -30 B. -15 C. -5 D. $-\frac{5}{6}$ E. $-\frac{1}{6}$

*7. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ converges?

A. $-\frac{5}{2}<x<-\frac{1}{2}$ B. $-\frac{5}{2}<x\le-\frac{1}{2}$ C. $-\frac{5}{2}\le x<-\frac{1}{2}$
D. $-\frac{1}{2}<x<-\frac{1}{2}$ E. $x\le-\frac{1}{2}$

*8. Which of the following is the Maclaurin series for e^{3x}?

A. $1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

B. $3+9x+\frac{27x^2}{2}+\frac{81x^3}{3!}+\frac{243x^4}{4!}+\cdots$

C. $1-3x+\frac{9x^2}{2}-\frac{27x^3}{3!}+\frac{81x^4}{4!}+\cdots$

D. $1+3x+\frac{3x^2}{2}+\frac{3x^3}{3!}+\frac{3x^4}{4!}+\cdots$

E. $1+3x+\frac{9x^2}{2}+\frac{27x^3}{3!}+\frac{81x^4}{4!}+\cdots$

*9. What is the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 2^n}$?

A. $1<x<5$ B. $1\le x<5$ C. $1\le x\le 5$ D. $2<x<4$ E. $2\le x\le 4$

*10. The third-degree Taylor polynomial for a function f about x = 4 is $\frac{(x-4)^3}{512}-\frac{(x-4)^2}{64}+\frac{(x-4)}{4}+2$.

What is the value of $f'''(4)$?

A. $-\frac{1}{64}$ B. $-\frac{1}{32}$ C. $\frac{1}{512}$ D. $\frac{3}{256}$ E. $\frac{81}{256}$

*11. Which of the following is the Maclaurin series for $\frac{1}{(1-x)^2}$?

A. $1-x+x^2-x^3+\cdots$ B. $1-2x+3x^2-4x^3+\cdots$ C. $1+2x+3x^2+4x^3+\cdots$

D. $1+x^2+x^4+x^6+\cdots$ E. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$

*12. Let f be the function given by $f(x)=\sin\left(5x+\frac{\pi}{4}\right)$, and let *P(x)* be the third-degree Taylor polynomial for f about $x = 0$.

a. Find *P(x)*.

b. Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

c. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{100}$.

d. Let *G* be the function given by $G(x)=\int_0^x f(t)dt$. Write the third-degree Taylor polynomial for *G* about $x = 0$.

*13. The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...+\frac{(-1)^n x^{2n}}{(2n+1)!}+... \text{ for all real numbers } x.$$

a. Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$.

b. Show that $1-\frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

c. Show that $y = f(x)$ is a solution to the differential equation $xy'+y=\cos x$