7. **2007 BC Exam - #6 – Form B (No Calculator)**

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $f(x) = 6e^{-x/3}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$6e^{-x/3} = 6\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{3}x\right)^n}{n!}$$

$$= 6\sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{3}\right)^n x^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{6(-1)^n x^n}{3^n \cdot n!}}$$ general term

$n=0$: $\dfrac{6(-1)^0 x^0}{3^0 \cdot 0!} = \dfrac{6(1)}{1} = 6$

$n=1$: $\dfrac{6(-1)^1 x^1}{3^1\, 1!} = \dfrac{-6x}{3} = -2x$

$n=2$: $\dfrac{6(-1)^2 x^2}{3^2 \cdot 2!} = \dfrac{6x^2}{18} = \dfrac{x^2}{3}$

$n=3$: $\dfrac{6(-1)^3 x^3}{3^3 \cdot 3!} = \dfrac{-6x^3}{162}$ $\dfrac{-x^3}{27}$

$$\boxed{6 - 2x + \frac{1}{3}x^2 - \frac{1}{27}x^3}$$

(b) $\int_0^x f(t)\,dt = \int_0^x 6 - 2t + \frac{1}{3}t^2 - \frac{1}{27}t^3\,dt$

$$6t - t^2 + \frac{t^3}{9} - \frac{t^4}{108}\Big]_0^x$$

$$\boxed{6x - x^2 + \frac{1}{9}x^3 - \frac{1}{108}x^4}$$

$$\int \sum_{n=0}^{\infty} \frac{6(-1)^n x^n}{3^n \cdot n!} = \boxed{\sum_{n=0}^{\infty} \frac{6(-1)^n x^{n+1}}{3^n \cdot n!\,(n+1)}}$$ General TERM

(c) $h(x) = Kf'(ax)$

$f(x) = 6e^{-x/3}$

$f'(x) = 6e^{-x/3}\left(-\frac{1}{3}\right) = -2e^{-x/3}$

$h(x) = K(-2e^{-ax/3})$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = e^x$

$-2Ke^{-ax/3} = e^x$

$-2K = 1$

$\boxed{K = -1/2}$

$-\frac{a}{3} = 1$

$-a = 3$

$\boxed{a = -3}$

8. 2008 BC Exam - #3 (Calculator Active)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) center → 2

$$h(2) + h'(2)(x-2)$$

$$\boxed{80 + 128(x-2)} \rightarrow h(1.9) \approx 80 + 128(1.9-2)$$

$$\approx \boxed{67.2}$$

Since $h''(x) > 0$, h is concave up on the interval. So the approximation is less than the actual value.

(b) $$h(2) + h'(2)(x-2) + \frac{h''(2)(x-2)^2}{2!} + \frac{h'''(2)(x-2)^3}{3!}$$

$$80 + 128(x-2) + \frac{488}{3}\frac{(x-2)^2}{2} + \frac{448}{3}\cdot\frac{(x-2)^3}{6}$$

$$\boxed{80 + 128(x-2) + \frac{244}{3}(x-2)^2 + \frac{224}{9}(x-2)^3}$$

$$h(1.9) \approx 80 + 128(1.9-2) + \frac{244}{3}(1.9-2)^2 + \frac{224}{9}(1.9-2)^3 = \boxed{67.988}$$

(c) Lagrange Error Bound $= \frac{h^4(c)}{4!}(2-1.9)^4$

$= \frac{\frac{584}{9}}{4!}(2-1.9)^4$

$\boxed{2.704 \times 10^{-4}} < 3.0 \times 10^{-4}$

9. 2009 BC Exam - #6 – Form B (No Calculator)

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty}(x+1)^n$$

for all real numbers x for which the series converges.

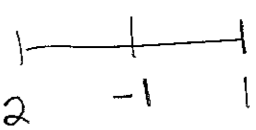
(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) $\sum_{n=0}^{\infty}(x+1)^n \quad \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{\cancel{(x+1)^n}}\right| = \lim_{n\to\infty}|(x+1)| = |x+1| < 1 \quad R=1$



geometric $\rightarrow$ $\boxed{(-2,1)}$

(b) $\text{sum} = \frac{a_1}{1-r} = \frac{1}{1-(x+1)} = \frac{1}{1-x-1} = \boxed{\frac{-1}{x}}$

(c) $g(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} -\frac{1}{t}\,dt = -\ln|t|\Big]_{-1}^{x}$

$= -\ln|x| + \ln|-1|$

$= -\ln|x|$

$g\left(-\frac{1}{2}\right) = -\ln\left|-\frac{1}{2}\right| = -\ln\frac{1}{2} = \ln\left(\frac{1}{2}\right)^{-1}$

$\boxed{\ln 2}$

(d) $h(x) = f(x^2-1) =$

$$1 + (x^2-1+1) + (x^2-1+1)^2 + \dots + (x^2-1+1)^n$$

$$= \boxed{1 + x^2 + x^4} + \dots + x^{2n}$$

$\sum_{n=0}^{\infty} (x^2)^n$ Geometric Series

$$\text{Sum} = \frac{1}{1-x^2}$$

$$h\left(\tfrac{1}{2}\right) = \frac{1}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}}$$

$$= \boxed{\frac{4}{3}}$$

10. 2010 BC Exam - #6 (No Calculator)

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$ general term

$\underline{n=0}$: $\frac{(-1)^0 x^{2(0)}}{(2\cdot 0)!} = 1$

$\underline{n=1}$: $\frac{(-1)^1 x^{2(1)}}{(2\cdot 1)!} = \frac{-x^2}{2}$

$\underline{n=2}$: $\frac{(-1)^2 x^{2(2)}}{(2\cdot 2)!} = \frac{x^4}{24}$

$\cos x = \boxed{1 - \frac{x^2}{2} + \frac{x^4}{24}}$

$\cos x - 1 = \frac{-x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$\boxed{\frac{\cos x - 1}{x^2} = \frac{-1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!}} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+2)!}}$ general term

0 1 2

(b) $f(x) = \frac{-1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots$

$= \frac{-1}{2} + \frac{1}{24}x^2 - \frac{1}{720}x^4 + \ldots$

$f'(x) = -0 + \frac{2}{24}x - \frac{4}{720}x^3 + \ldots \rightarrow f'(0) = 0$

critical point at $x = 0$

$f''(x) = \frac{1}{12} - \frac{12}{720}x^2 + \ldots \quad f''(0) = \frac{1}{12}$

positive (concave up)

$\boxed{f(x) \text{ has a relative maximum at } x=0 \text{ b/c } f'(0)=0 \text{ \& } f''(0)>0}$

(c) $g(x) = 1 + \int_0^x f(t)\,dt = 1 + \int_0^x -\frac{1}{2} + \frac{1}{24}t^2 - \frac{1}{720}x^4 + \ldots \, dt$

$-\frac{1}{2}t + \frac{1t^3}{24 \cdot 3} - \frac{t^5}{720 \cdot 5}\Big]_0^x$

$\boxed{1 - \frac{1}{2}x + \frac{x^3}{24 \cdot 3} - \frac{x^5}{720 \cdot 5}}$

(d) $g(1) = 1 - \frac{1}{2}(1) - \frac{(1)^3}{24 \cdot 3}$

$= \boxed{1 - \frac{1}{2} - \frac{1}{24 \cdot 3}}$

use 4th term to find estimation of error

$\frac{+x^5}{720 \cdot 5} = \frac{+(1)^5}{720 \cdot 5}$

$\boxed{\frac{+1}{720 \cdot 5} = \frac{+1}{6! \cdot 5} < \frac{1}{6!}}$