AP CALCULUS BC NAME Answer Key *
Unit 10 – Free Response Questions (Taylor Series)

1. 2007 BC Exam - #6 (No Calculator)

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$. center = 0

(b) Use your answer to part (a) to find $\lim\limits_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\frac{1}{200}$.

(a) $f(x) = e^{-x^2}$ $\quad f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n})}{n!}$$

$n=0$: $\dfrac{(-1)^0(x^{2(0)})}{0!} = 1$

$n=1$: $\dfrac{(-1)^1(x^{2(1)})}{1!} = -x^2$

$n=2$: $\dfrac{(-1)^2(x^{2(2)})}{2!} = \dfrac{x^4}{2}$

$n=3$: $\dfrac{(-1)^3(x^{2(3)})}{3!} = \dfrac{-x^6}{6}$

$$\boxed{1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6}$$

$$\boxed{\text{General term: } \frac{(-1)^n(x^{2n})}{n!}}$$

(b) $$\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4}$$

$$\lim_{x \to 0} \frac{\cancel{1} \cancel{-x^2} - (\cancel{1} \cancel{-x^2} + \frac{1}{2}x^4 - \frac{1}{6}x^6)}{x^4}$$

$$\lim_{x \to 0} \frac{-\frac{1}{2}x^4 + \frac{1}{6}x^6}{x^4} \to \lim_{x \to 0} -\frac{1}{2} + \frac{1}{6}x^2 = \boxed{-\frac{1}{2}}$$

(c) $$\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 + \dots + \frac{(-1)^n t^{2n}}{n!} + \dots\right)dt$$

$$= t - \frac{1}{3}t^3 + \frac{1}{10}t^5 - \frac{1}{42}t^7 + \dots \Big]_0^x$$

$$= \boxed{x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \dots}$$

$$\int_0^{1/2} e^{-t^2}\,dt = t - \frac{1}{3}t^3 \Big]_0^{1/2}$$

$$\left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3\right) - \left(0 - \frac{1}{3}(0)^3\right)$$

$$\frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right)$$

$$\frac{1}{2} - \frac{1}{24}$$

$$\frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

(d) $\int_0^{-1/2} e^{-t^2}$ is alternating series whose terms decrease in absolute value to 0.

→ use the 3rd term: $\frac{1}{10} t^5$

$\frac{1}{10}\left(\frac{1}{2}\right)^5 = \frac{1}{10}\left(\frac{1}{32}\right)$

$= \boxed{\frac{1}{320} < \frac{1}{200}}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$$

$$\frac{\left(\frac{1}{2}\right)^5}{2!(5)} = \left(\frac{1}{2}\right)^5 \left(\frac{1}{10}\right)$$

2. 2006 BC Exam - #6 (No Calculator)

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

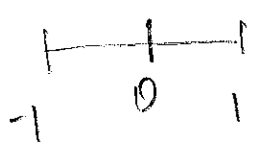
(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1}$ center $= 0$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(n+1)x^{n+1}}{(n+2)} \cdot \frac{(n+1)}{\cancel{(-1)^n} n \cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{x(n+1)^2}{n(n+2)} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{(n+1)^2}{n(n+2)} \right| = |x|$$

$|x| < 1$ $R = 1$ radius $= 1$



$x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n n(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{n}{n+1}$ $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ diverges → nth term test

$x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n n(1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{n+1}$ Alt. Series test $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ diverges nth term test

$$\boxed{(-1, 1)}$$

(b) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n+1}$ $\qquad$ $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

$\downarrow$ $\qquad$ $\downarrow$

$-\frac{1}{2}x + \frac{2}{3}x^2 - \frac{3}{4}x^3 + \ldots$ $\qquad$ $1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \ldots$

$f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \ldots$ $\qquad$ $f'(0) = -\frac{1}{2}$

$f''(x) = 0 + \frac{4}{3} - \frac{18}{4}x + \ldots$ $\qquad$ $f''(0) = \frac{4}{3}$

$g'(x) = 0 - \frac{1}{2} + \frac{2}{24}x - \frac{3}{720}x^2 + \ldots$ $\qquad$ $g'(0) = -\frac{1}{2}$

$g''(x) = 0 + \frac{2}{24} - \frac{6}{720}x + \ldots$ $\qquad$ $g''(0) = \frac{1}{12}$

$y'(0) = f'(0) - g'(0) = -\frac{1}{2} - (-\frac{1}{2}) = \boxed{0}$

$y''(0) = f''(0) - g''(0) = \frac{4}{3} - \frac{1}{12} = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \boxed{\frac{5}{4}}$

Since $\underbrace{y'(0) = 0}_{\text{shows critical point at } 0.}$ and $y''(0) > 0$, then y has a relative minimum at $x = 0$.

3. 2005 BC Exam - #6 (No Calculator)

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

center

(a) Write the sixth-degree Taylor polynomial for f about $\boxed{x = 2.}$

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $$\overset{0}{f(2)} + \overset{1}{\cancel{f'(2)(x-2)}} + \overset{2}{\frac{f''(2)(x-2)^2}{2!}} + \overset{3}{\cancel{\frac{f'''(2)(x-2)^3}{3!}}}$$

$$+ \overset{4}{\frac{f^4(2)(x-2)^4}{4!}} + \overset{5}{\cancel{\frac{f^5(2)(x-2)^5}{5!}}} + \overset{6}{\frac{f^6(2)(x-2)^6}{6!}}$$

$$\overset{0}{7} + \overset{2}{\frac{(2-1)!}{3^2} \cdot \frac{(x-2)^2}{2!}} + \overset{4}{\frac{(4-1)!}{3^4} \cdot \frac{(x-2)^4}{4!}} + \overset{6}{\frac{(6-1)!}{3^6} \cdot \frac{(x-2)^6}{6!}}$$

$$\boxed{\overset{0}{7} + \overset{1}{\frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2} + \overset{2}{\frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4} + \overset{3}{\frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6}}$$

(b) $$\sum \frac{(2n-1)!(x-2)^{2n}}{3^{2n}(2n)!} = \frac{(2n-1)!}{3^{2n}(2n)!} = \frac{\cancel{(2n-1)!}}{3^{2n}(2n)\cancel{(2n-1)!}} = \boxed{\frac{1}{3^{2n}(2n)}}$$

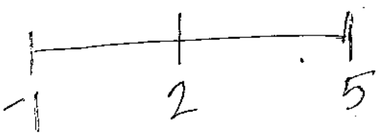
2(n+1)

(c) $7 + \sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{3^{2n}(2n)}$ center = 2

$$\lim_{n\to\infty} \left| \frac{(x-2)^{2n+2}}{3^{2n+2}(2n+2)} \cdot \frac{\cancel{3^{2n}}(2n)}{\cancel{(x-2)^{2n}}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^2(2n)}{3^2(2n+2)} \right| = \left| \frac{(x-2)^2}{9} \right| \lim_{n\to\infty} \left| \frac{2n}{2n+2} \right| = \left| \frac{(x-2)^2}{9} \right|$$

$$\left| \frac{(x-2)^2}{9} \right| < 1$$



$|(x-2)^2| < 9$ $\quad |x-2| < 3$

radius = 3

$\underline{x=-1}$: $\quad 7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{3^{2n}(2n)} = 7 + \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n}$

Alt Series Test $\lim_{n\to\infty} \frac{1}{2n} = 0$ ✓

$\frac{1}{2n+2} \le \frac{1}{2n}$ ✓

converges

$\underline{x=5}$: $\quad 7 + \sum_{n=1}^{\infty} \frac{(3)^{2n}}{3^{2n}(2n)} = 7 + \sum_{n=1}^{\infty} \frac{1}{2n} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$

p-series test
diverges

$\boxed{[-1,5)}$

4. 2000 BC Exam - #3 (No Calculator)

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $\boxed{x = 5.}$ center

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a)

$$\overset{0}{f(5)} + \overset{1}{f'(5)(x-5)} + \overset{2}{\frac{f''(5)(x-5)^2}{2!}} + \overset{3}{\frac{f'''(5)(x-5)^3}{3!}}$$

$$\frac{1}{2} + \underbrace{\frac{(-1)^1(1)!}{2^1(1+2)}}_{\frac{-1}{6}}(x-5) + \underbrace{\frac{(-1)^2 \cancel{2!}}{2^2(2+2)}}_{\frac{1}{4(4)}} \frac{(x-5)^2}{\cancel{2!}} + \underbrace{\frac{(-1)^3 \cancel{(3)!}}{2^3(3+2)}}_{\frac{-1}{8(5)}} \cdot \frac{(x-5)^3}{\cancel{3!}}$$

$$\boxed{\frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3}$$

(b)

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{2^n(n+2)}$$

n! canceled

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}(x-5)^{\cancel{n+1}}}{2^{n+1}(n+3)} \cdot \frac{\cancel{2^n}(n+2)}{\cancel{(-1)^n}\,\cancel{(x-5)^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)(n+2)}{2(n+3)}\right|$$

$$\left|\frac{x-5}{2}\right| \lim_{n\to\infty}\left|\frac{n+2}{n+3}\right| = \left|\frac{x-5}{2}\right| \qquad \left|\frac{x-5}{2}\right| < 1$$

$$|x-5| < 2 \quad \boxed{\text{radius} = 2}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{2^n(n+2)}$ is alternating series whose terms decrease to absolute value of 0.

use 7th term (next term) $\frac{1}{2^7(7+2)} = \frac{1}{2^7(9)} = \boxed{\frac{1}{1152} < \frac{1}{1000}}$

5. 2015 BC Exam - #6 (No Calculator)

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

(a) $\lim_{n\to\infty} \left| \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} \right| = \lim_{n\to\infty} \left| \frac{3(x)(n)}{(n+1)} \right|$

$|3x| \cdot \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |3x| \qquad |3x| < 1 \qquad |x| < \frac{1}{3}$

$\boxed{R = 1/3}$

(b) $f(x) = x - \frac{3}{2}x^2 + 3x^3 + \frac{(-3)^{4-1}}{4}x^4$

$-\frac{27}{4}x^4$

$\boxed{f'(x) = 1 - 3x + 9x^2 - 27x^3}$

$\sum_{n=0}^{\infty} (-3)^n x^n = \sum_{n=0}^{\infty} (-3x)^n \leftarrow$ Geometric Series

$a_1 = 1$

$r = -3x$

$\boxed{f'(x) = \frac{1}{1+3x}}$

(c) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\boxed{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$$

$$g(x) = e^x f(x) = \left(1 + x + \frac{1}{2}x^2 + \dots\right)\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right)$$

$$1\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right) + x\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right) + \frac{1}{2}x^2\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right)$$

$$x - \frac{3}{2}x^2 + 3x^3 + \dots + x^2 - \frac{3}{2}x^3 + 3x^4 + \dots + \frac{1}{2}x^3 - \frac{3}{4}x^4 + \frac{3}{2}x^5 + \dots$$

$$x + \left(-\frac{3}{2} + 1\right)x^2 + \left(3 - \frac{3}{2} + \frac{1}{2}\right)x^3 + \dots$$

$$\boxed{x - \frac{1}{2}x^2 + 2x^3}$$

6. 2014 BC Exam - #6 (No Calculator)

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

ⓐ $\lim_{n\to\infty}\left| \frac{\cancel{(-1)^{n+2}}\, 2^{n+1}\,(x-1)^{n+1}}{(n+1)} \cdot \frac{n}{\cancel{(-1)^{n+1}}\,\cancel{2^n}\,\cancel{(x-1)^n}} \right|$

$\lim_{n\to\infty}\left| \frac{2(x-1)(n)}{n+1} \right| = |2(x-1)| \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |2(x-1)|$

$|2(x-1)| < 1$

$|x-1| < \frac{1}{2}$ $\boxed{R = 1/2}$

ⓑ $f(x) = \frac{(-1)^{1+1}\, 2^1\,(x-1)^1}{1} + \frac{(-1)^{2+1}\, 2^2 (x-1)^2}{2} + \frac{(-1)^{3+1}\, 2^3 (x-1)^3}{3}$

$(-1)^2 2(x-1)$ $\quad \frac{(-1)^3(4)}{2}$ $\quad \frac{(-1)^4 8}{3}$

$f(x) = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 + \ldots$

$\boxed{f'(x) = 2 - 4(x-1) + 8(x-1)^2}$

1 2 3

general term = $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}}$

(1) $\sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}$

$a_1 = 2$ (first term of $f'(x)$)

$r = -2(x-1)$

$f'(x) = \frac{2}{1+2(x-1)} = \frac{2}{1+2x-2} = \boxed{\frac{2}{2x-1}}$

$\int f'(x) = \int \frac{2}{2x-1}$

$u = 2x-1$

$du = 2dx$

$\int \frac{du}{u}$

$\ln|2x-1| + C$

$f(1) = 0$ b/c

$f(1) = 2(1-1) - 2(1-1)^2 + \ldots$

$f(1) = 0$

$0 = \ln|2(1)-1| + C$

$0 = \ln 1 + C$

$0 = 0 + C$

$C = 0$

$\boxed{f(x) = \ln|2x-1|}$