AP Calculus BC
Unit 10 – Sequences & Series (Part 2)

Day 6 Notes: Taylor Series (Part 1)

Every convergent power series must take the form $\sum_{n=0}^{\infty} a_n (x-c)^n$. For Taylor series, $a_n = \frac{f^{(n)}(c)}{n!}$

A **Taylor series** will be in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

A **Maclaurin series** is simply a special case of Taylor series where c = 0.

$c=0$

Example 1: Find a Maclaurin series for $f(x) = e^{-2x}$. Then find its interval of convergence.

$f(x) = e^{-2x} \to e^0 = 1$

$f'(x) = -2e^{-2x} \to -2$

$f''(x) = 4e^{-2x} \to 4$

$f'''(x) = -8e^{-2x} \to -8$

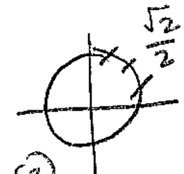
$f^4(x) = 16e^{-2x} \to 16$

0 1 2 3 4

$$1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots$$

$$1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2x^4}{3} + \cdots$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^n}{n!}}$$



Example 2: Use the definition to find a Taylor series for $f(x) = sinx$, centered at $c = \frac{\pi}{4}$.

$f(x) = \sin x \to \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \quad \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \quad -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \quad -\frac{\sqrt{2}}{2}$

$f^4(x) = \sin x \quad \frac{\sqrt{2}}{2}$

0 1 2 3 4

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^3}{3!} + \frac{\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^4}{4!} + \cdots$$

$$(-1)^{n(n+1)/2} = 1, 1, -1, -1, 1, 1, \ldots$$

$$\boxed{\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} \left(x-\frac{\pi}{4}\right)^n}{n!}}$$

5 POWER SERIES TO KNOW

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
2. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$
3. $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n+1}(x-1)^n}{n} + \cdots$
4. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$
5. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$

You can use these known power series to generate power series for related functions! This really saves time.

Example 3: Find the Maclaurin series for $f(x) = e^{-3x}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (x)^n}{n!}}$$

Example 4: Find a power series for $f(x) = \frac{1}{1-x^2}$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (x^2)^n = \boxed{\sum_{n=0}^{\infty} x^{2n}}$$

Example 5: Find a power series for $f(x) = \cos x^3 - 3x^2$.

$\swarrow$ # or Polynomial

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{6n})}{(2n)!}$$

$$\boxed{-3x^2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}}$$