AP Calculus BC
Unit 10 – Sequences & Series (Part 2)

Day 5 Notes: Geometric Power Series

We can write a power series for some functions in the form of a geometric series

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, where $|r|<1$. We may have to manipulate $f(x)$ to put it in the form $\frac{a}{1-r}$.

$$S = \frac{a}{1-r}$$

Examples:

1. Write a geometric power series centered at $c = 0$ for $f(x) = \frac{3}{2x-1}$.

$$\frac{3}{2x-1} = \frac{3}{-1+2x} = \frac{-3}{1-2x} \qquad a=-3 \quad r=(2x)$$

$$\boxed{\sum_{n=0}^{\infty} (-3)(2x)^n}$$

2. Find a geometric power series centered at c = -2 for $f(x) = \frac{3}{4-x}$.

**Take care of the center first!

$$\frac{3}{4-x} \rightarrow \frac{3}{4-(x+2)+2} = \frac{3}{6-(x+2)} = \frac{3/6}{\frac{6}{6}-\frac{(x+2)}{6}} = \frac{1/2}{1-\frac{(x+2)}{6}} \qquad a=\frac{1}{2} \quad r=\frac{(x+2)}{6}$$

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+2}{6}\right)^n}$$

3. Write a geometric power series centered at $c = 0$ for $f(x) = \frac{4x-7}{2x^2+3x-2}$. Partial Fractions

$$\frac{4x-7}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$$

$4x-7 = A(2x-1) + B(x+2)$

$x=\frac{1}{2}$: $4(\frac{1}{2})-7 = B(\frac{1}{2}+2)$

$B=-2$

$x=-2$: $4(-2)-7 = A(2(-2)-1)$

$A=3$

$$\frac{3}{x+2} + \frac{-2}{2x-1}$$

- $\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+x/2} \qquad a=3/2 \quad r=-x/2 \qquad \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{x}{2}\right)^n$
- $\frac{-2}{2x-1} = \frac{-2}{-1+2x} = \frac{2}{1-2x} \qquad a=2 \quad r=(2x) \qquad \sum_{n=0}^{\infty}(2)(2x)^n$

$$\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{x}{2}\right)^n + \sum_{n=0}^{\infty}(2)(2x)^n = \boxed{\sum_{n=0}^{\infty}\left[\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)^n + (2)(2^n)\right]x^n}$$

Convergence of Geometric Power Series

Let $\sum_{n=0}^{\infty} ar^n$ be a geometric power series.

- A geometric series will converge when $|r| < 1$.
- If a geometric series is centered at $x = c$ and the radius of convergence is R, the interval of convergence is $(c - R, c + R)$.
- A geometric series will never converge at the endpoints of the interval of convergence.

(,)

Examples:

4. Find a power series for $f(x) = \frac{4}{3x+2}$ centered at $\boxed{c = 2}$. Then find the interval of convergence.

$$\frac{4}{3x+2} \rightarrow \frac{4}{2+3x} \rightarrow \frac{4}{2+3(x-2)+6} = \frac{4}{8+3(x-2)} = \frac{4/8}{\frac{8}{8}+\frac{3(x-2)}{8}} = \frac{1/2}{1+\frac{3(x-2)}{8}}$$

$a = 1/2$

$r = \frac{-3(x-2)}{8}$

$$\boxed{\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)\left(\frac{-3(x-2)}{8}\right)^n}$$

$$\left|\frac{-3(x-2)}{8}\right| < 1$$

$$|x-2| < \frac{8}{3} \qquad \boxed{R = \frac{8}{3}}$$

$$\boxed{\text{interval: } \left(2-\tfrac{8}{3},\, 2+\tfrac{8}{3}\right)}$$

5. Write a geometric power series centered at $c = 0$ for $f(x) = \frac{3x-1}{x^2-1}$. Then find the interval of convergence.

$$\frac{3x-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$3x-1 = A(x-1) + B(x+1)$

$x=1$: $3(1)-1 = B(1+1)$

$B = 1$

$x=-1$: $3(-1)-1 = A(-1-1)$

$A = 2$

$$\frac{2}{x+1} + \frac{1}{x-1}$$

$$\frac{2}{x+1} = \frac{2}{1+x} \qquad a = 2, \quad r = -x \qquad \sum_{n=0}^{\infty} (2)(-x)^n$$

$$\frac{1}{x-1} = \frac{1}{-1+x} = \frac{-1}{1-x} \qquad a = -1, \quad r = x \qquad \sum_{n=0}^{\infty} (-1)(x)^n$$

$$\sum_{n=0}^{\infty} (2)(-x)^n + \sum_{n=0}^{\infty} (-1)(x^n) = \sum_{n=0}^{\infty} \left[(2)(-1)^n - 1\right] x^n$$

$$|x| < 1 \qquad \boxed{R = 1}$$

-1 0 1

$$\boxed{(-1, 1)}$$