AP Calculus BC
Unit 10 – Sequences & Series (Part 2)

Day 4 Notes: Power Series (Part 2)

Properties of Functions Defined by Power Series

If $f(x)=a_n(x-c)^n$ has a radius of convergence R > 0, then on the interval $(c-R,\ c+R)$, f is continuous and differentiable.

1. $f'(x)=\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$

2. $\int f(x)\,dx = C+\sum a_n\cdot\frac{(x-c)^{n+1}}{n+1}$

***The radius of convergence is the same for $f(x)$, $f'(x)$, and $\int f(x)\,dx$.
However, the interval of convergence may differ at the endpoints. **Always check for converge at the endpoints of the interval!**

Example 1: Given $f(x)$, find $f'(x)$ and $\int f(x)\,dx$

a) $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$ $\frac{1}{n}x^n$

$\boxed{f'(x)=\sum_{n=1}^{\infty} x^{n-1}}$

$\boxed{\int f(x)dx=\sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)}}$

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-4)^n}{n}$ $\frac{1}{n}(x-4)^n$

$\boxed{f'(x)=\sum_{n=1}^{\infty}(-1)^{n+1}(x-4)^{n-1}}$

$\boxed{\int f(x)dx=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-4)^{n+1}}{n(n+1)}}$

c) $\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$

$f'(x)=\sum_{n=1}^{\infty} n\left(\frac{x}{3}\right)^{n-1}\left(\frac{1}{3}\right) = \boxed{\sum_{n=1}^{\infty}\frac{n}{3}\left(\frac{x}{3}\right)^{n-1}}$

$\int f(x)dx=\sum_{n=1}^{\infty}\frac{\left(\frac{x}{3}\right)^{n+1}}{(n+1)}(3) = \boxed{\sum_{n=1}^{\infty}\left(\frac{3}{n+1}\right)\left(\frac{x}{3}\right)^{n+1}}$

$\frac{1}{n}(2x)^n$

Example 2: Find the intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$ if

$$f(x)=\sum_{n=1}^{\infty}\frac{(2x)^n}{n} \qquad C=0$$

$$\lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)}\cdot\frac{(n)}{\cancel{(2x)^n}}\right| = \lim_{n\to\infty}\left|\frac{(2x)(n)}{(n+1)}\right| = |2x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |2x|$$

$|2x|<1 \to |x|<\frac{1}{2}$

$R=1/2$

$-1/2 \quad 0 \quad 1/2$

f(x)

$x=-1/2$: $\sum_{n=1}^{\infty}\frac{(2\cdot -\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Alt. Series Test

$\lim_{n\to\infty}\frac{1}{n}=0$ ✓ $\frac{1}{n+1}\le\frac{1}{n}$ ✓

converges

$x=1/2$: $\sum_{n=1}^{\infty}\frac{(2\cdot\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ p-series test

$p=1$ diverges

$\boxed{f(x)\to[-1/2,1/2)}$

$f'(x)=\sum_{n=1}^{\infty}2(2x)^{n-1}$

$x=-1/2$: $\sum_{n=1}^{\infty}2(2\cdot -\frac{1}{2})^{n-1} = \sum_{n=1}^{\infty}2(-1)^{n-1}$

Alt. Series Test

$\lim_{n\to\infty}2\neq0$ Diverges by nth term

$x=1/2$: $\sum_{n=1}^{\infty}2(2\cdot\frac{1}{2})^{n-1}$

$\sum_{n=1}^{\infty}2(1)^{n-1}$ Geometric Series $r=1$, diverges or $\sum_{n=1}^{\infty}2 = 2+2+2+2\ldots$ diverges

$\boxed{f'(x)\to(-1/2,1/2)}$

$\frac{1}{n}(2x)^n$

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{1\,(2x)^{n+1}}{2n(n+1)} = \sum_{n=1}^{\infty} \frac{(2x)^{n+1}}{2n(n+1)}$$

$\underline{x = -1/2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2\cdot -\frac{1}{2})^{n+1}}{2n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n(n+1)}$

Alt. Series Test

$\displaystyle\lim_{n\to\infty} \frac{1}{2n(n+1)} = 0$ ✓

$\displaystyle\frac{1}{(2n+2)(n+2)} \le \frac{1}{2n(n+1)}$ ✓

converges

$\underline{x = 1/2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2\cdot 1/2)^{n+1}}{2n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)}$

Telescoping Series

converges

$$\boxed{\int f(x)\,dx \rightarrow [-1/2, 1/2]}$$

$(-1)^{n+1}(-1)^n$ $\quad$ $\underline{n=1}$ $\quad (-1)^2(-1)^1 = -1$

$\underline{n=2} \quad (-1)^3(-1)^2 = -1$

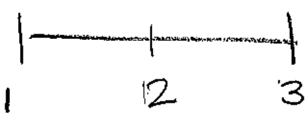
$\underline{n=3} \quad (-1)^4(-1)^3 = -1$

Example 3: Find the intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$ if

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n} \qquad c = 2 \qquad \frac{1}{n}(x-2)^n$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}(x-2)^{n+1}}{(n+1)} \cdot \frac{(n)}{\cancel{(-1)^{n+1}}\cancel{(x-2)^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)(n)}{(n+1)} \right| = |x-2| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x-2|$$

$$|x-2| < 1 \qquad R = 1$$



$\underline{x=1}$: $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n} \frac{-1}{n}$ p-series test diverges

$\underline{x=3}$: $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^n}{n} = \sum_{n=1} \frac{(-1)^n(-1)^1(1)^n}{n}$

$\displaystyle = \sum_{n=1}^{\infty} \frac{-1(-1)^n}{n}$ Alt Series Test

$\displaystyle \lim_{n\to\infty} \frac{1}{n} = 0$ ✓ $\qquad \frac{1}{n+1} \le \frac{1}{n}$ ✓

Converges

$\boxed{f(x) \to (1, 3]}$

$\displaystyle f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}$ $\qquad$ $\underline{x=1}$: $\displaystyle \sum_{n=1}^{\infty} (-1)^{n+1}(1-2)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1}(-1)^{n-1}$

$\displaystyle \sum_{n=1}^{\infty} (-1)^n(-1)^1(-1)^n(-1)^{-1} = \sum_{n=1}^{\infty} (-1)^1(-1)^{-1}$

$\displaystyle \sum_{n=1}^{\infty} 1 = 1+1+1+1\ldots$

diverges

↓

Unit 10 – Day 4 Assignment:

Find the intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$ if…

1) $\displaystyle f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ $\qquad$ 2) $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ $\qquad$ 3) $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$\underline{x=3}$: $\sum_{n=1}^{\infty} (-1)^{n+1} (3-2)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} (1)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n}(-1)^{1}(1)^{n}(1)^{-1}$

$= \sum_{n=1}^{\infty} (-1)^{n}(-1)$ Alt Series Test

$\lim_{n\to\infty} -1 \neq 0$ diverg by nth term

$$\boxed{f'(x) \to (1,3)}$$

$\int f(x)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}$

$\underline{x=1}$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1-2)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{n+1}}{n(n+1)}$ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Telescoping Series converges

$\underline{x=3}$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3-2)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$

Alt. Series Test

$\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$ ✓

$\frac{1}{(n+1)(n+2)} \leq \frac{1}{n(n+1)}$ ✓

converges

$$\boxed{\int f(x)dx \to [1,3]}$$