AP Calculus BC
Unit 10 – Sequences & Series (Part 2)

Day 3 Notes: Power Series (Part 1)

Recall that Maclaurin and Taylor polynomials are <u>finite</u> polynomials that can be used to <u>approximate</u> a function $f(x)$. For example, we found that $f(x) = e^x$ can be approximated by $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$. The higher the degree of the polynomial, the better the approximation.We can go further, because $f(x) = e^x$ can be represented **<u>exactly</u>** by the **POWER SERIES** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

<u>Definition of power series:</u>
Let a_n be a constant and x be a variable.
1) A power series centered at $x = 0$ takes the form
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$
2) A power series centered at $x = c$ takes the form
$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$
c = Taylor | Maclaurin Series

A power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is a function of x, and its domain is the set of all x for which the series converges. **A series will always converge at its center!**

<u>Convergence of a power series centered at x = c</u>
One of the following will be true:
1. The series converges only at $x = c$. This means that the radius of convergence is 0. ($R = 0$) ∞ from Ratio test
2. There exists a number R, $R > 0$, such that

   $|x - c| < R \Rightarrow$ the series converges
   $|x - c| > R \Rightarrow$ the series diverges
   - R is called the **<u>radius of convergence</u>**.
   - The set of all x for which the power series converges is called the **<u>interval of convergence</u>**.
3. The series converges absolutely for all x. This means the radius of convergence is ∞. 0 from Ratio test

We use the **Ratio Test** to determine the radius of convergence R.

Examples:

1) At what point is each series centered?

a. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ $\boxed{C=0}$

b. $\sum_{n=0}^{\infty} 3(x+4)^n$ $\boxed{C=-4}$

2) Find the radius of convergence (R).

a. $\sum_{n=0}^{\infty} (4x)^n$ $C=0$

$$\lim_{n\to\infty}\left|\frac{(4x)^{\cancel{n+1}}}{\cancel{(4x)^n}}\right| = \lim_{n\to\infty}|4x|$$

$$= |4x|$$

$$|4x|<1$$

$$|x|<\frac{1}{4} \quad R=\frac{1}{4}$$

b. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$ $C=2$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x-2)}^{n+2}}{(n+2)3^{\cancel{n+2}}}\cdot\frac{(n+1)3^{\cancel{n+1}}}{\cancel{(x-2)^{n+1}}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-2)\ (n+1)}{(n+2)\ 3}\right| = \frac{|x-2|}{3}\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$= \frac{|x-2|}{3}<1$$

$$|x-2|<3$$

$$R=3$$

c. $\sum_{n=0}^{\infty} n!x^n$ $C=0$

$$\lim_{n\to\infty}\left|\frac{\overset{(n+1)(\cancel{n!})}{\cancel{(n+1)!}}\cdot x^{\cancel{n+1}}}{\cancel{n!}\cdot\cancel{x^n}}\right|$$

$$\lim_{n\to\infty}\left|n+1(x)\right| = |x|\lim_{n\to\infty}(n+1)$$

$$=\infty$$

$$R=0$$

$\downarrow$

converges only at center

d. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $C=0$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\, x^{\cancel{2n+3}\ 2}}{\underset{(2n+3)(2n+2)\cancel{(2n+1)!}}{(2n+3)!}}\cdot\frac{\cancel{(2n+1)!}}{\cancel{(-1)^n}\cancel{(x^{2n+1})}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right|$$

$$|x^2|\lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\right| = 0$$

$\downarrow$

$R=\infty$

converges for all x

The interval of convergence of a power series includes the interval $(c-R,\ c+R)$. **We must test the endpoints separately to determine if they should be included in the interval.**

3) Find the interval of convergence:

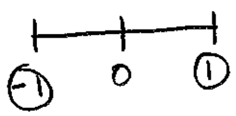
a. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$ $c=0$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}x^{n+1}}{\cancel{(n+2)}(n+3)}\cdot\frac{(n+1)\cancel{(n+2)}}{\cancel{(-1)^n}\cancel{x^n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x(n+1)}{(n+3)}\right| = |x|\lim_{n\to\infty}\left|\frac{n+1}{n+3}\right|$$

$|x|<1$

$R=1$



$\underline{x=-1}$ $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{(n+1)(n+2)} = \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}$ converges by Telescoping Series

$\underline{x=1}$: $\sum_{n=0}^{\infty}\frac{(-1)^n(1)^n}{(n+1)(n+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$

$a_n = \frac{1}{(n+1)(n+2)}$

$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0$ ✓

$\frac{1}{(n+2)(n+3)} \le \frac{1}{(n+1)(n+2)}$ ✓

convg. by Alt. Series

$\boxed{[-1,1]}$

b. $\sum_{n=0}^{\infty}\frac{(3x)^n}{(2n)!}$ $c=0$

$$\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+2)!}\cdot\frac{\cancel{(2n)!}}{(3x)^n}\right|$$

$(2n+2)(2n+1)\cancel{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right|$$

$$|3x|\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+1)}\right|$$

$=0$ $R=\infty$

$\boxed{(-\infty,\infty)}$

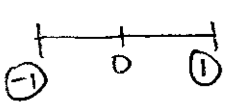
c. $\sum_{n=1}^{\infty}\frac{x^n}{n}$ $C=0$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{\cancel{x^n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x(n)}{(n+1)}\right|$$

$$|x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$ $|x|<1$

$R=1$



$\underline{x=-1}$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Alt. series conv $\lim_{n\to\infty}\frac{1}{n}=0$ ✓ $\frac{1}{n+1}\le\frac{1}{n}$ ✓

$\underline{x=1}$: $\sum_{n=1}^{\infty}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ p-series diverges

$\boxed{[-1,1)}$