AP Calculus BC
Unit ~~9~~ 10 – Sequences & Series (Part 2)

Day 2 Notes: Taylor Theorem

Remainder of a Taylor Polynomial

If $f(x)$ is approximated by a Taylor polynomial $P_n(x)$, then

$$\underbrace{f(x)}_{\text{exact}} = \underbrace{P_n(x)}_{\text{approx.}} + \underbrace{R_n(x)}_{\text{remainder}}$$

Therefore, the error in the approximation $P_n(x)$ is

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|$$

TAYLOR'S THEOREM

If f has n derivatives in an interval containing $x = c$, then for each x in that interval, there is a number z, between x and c, such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \quad \leftarrow \textbf{Lagrange form of remainder}$$

max value on [c, x]

Note: If Error = $|R_n(x)|$, then all we need is the **maximum value** of $f^{(n+1)}(z)$ on the interval from x to c. **We don't actually have to find the value of z!**

Example 1: Use Taylor's Theorem to obtain an upper bound for the error (remainder) of the approximation. Then calculate the exact value of the error.

$P_5(x)$
$c = 0$

$$e^{1} \approx \underbrace{1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}}_{\text{Taylor polynomial of degree 5 w/ 1 plugged in w/ } c=0}$$

$f(x) = e^x \quad f(1)$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$R_5(x) = \frac{f^{(6)}(z)}{6!}(x)^6$$

max on $[c, x] = [0, 1]$
e^1

$$R_5(x) = \frac{e^1}{6!}(1)^6 = \frac{e}{720} = 0.00378$$

$$\boxed{R_5(x) \leq 0.00378}$$

Exact error:

$e^1 = 2.718281828$

$1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} = 2.716666667$

$2.718281828 - 2.716666667 = 0.001615161$

Example 2: Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function sin(0.75) to be less than 0.001.

$f(x) = \sin x$

$$R_n(x) \le \frac{\overbrace{f^{n+1}(z)}^{\max = 1}}{(n+1)!}(0.3)^{n+1}$$

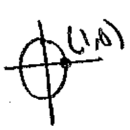
$$\frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

$n=1$: $\frac{1}{(1+1)!}(0.3)^{1+1} = 0.045$

$n=2$: $\frac{1}{(2+1)!}(0.3)^{2+1} = .0045$

$n=3$: $\frac{1}{(3+1)!}(0.3)^{3+1} = .0003375$

$\boxed{n=3}$



Example 3: Given f(x) = cosx.

a) Write a 4th degree Taylor polynomial for f(x) about x = 0.

$f(x) = \cos x \rightarrow \cos(0) = 1$
$f'(x) = -\sin x \rightarrow -\sin(0) = 0$
$f''(x) = -\cos x \rightarrow -\cos(0) = -1$
$f'''(x) = \sin x \rightarrow \sin(0) = 0$
$f^4(x) = \cos x \rightarrow \cos(0) = 1$

$$P_4(x) = 1 + \cancel{0x} + \frac{-1x^2}{2!} + \cancel{\frac{0x^3}{3!}} + \frac{1x^4}{4!}$$

$$\boxed{P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4}$$

b) Use the polynomial you found in part (a) to approximate the value of cos(0.2).

$$\cos(0.2) = 1 - \frac{1}{2}(0.2)^2 + \frac{1}{24}(0.2)^4 = \boxed{0.9800666667}$$

c) Use Taylor's Theorem to estimate the maximum error in your approximation.

$$R_4(x) = \frac{\overbrace{f^5(z)}^{\max \text{ on } [c,x] = [0,0.2] = 1}}{5!}(x-0)^5$$

$$= \frac{1}{5!}(0.2)^5 = 2.667 \times 10^{-6}$$

$$\boxed{R_4(x) \le 2.667 \times 10^{-6}}$$