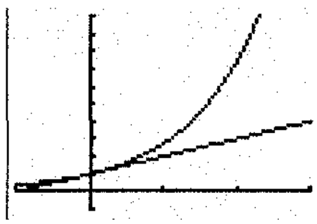
AP Calculus BC
Unit 9 – Sequences & Series (Part 2)

Day 1 Notes: Maclaurin & Taylor Polynomials

Recall: We can use the equation of the tangent to a curve at $x = c$ to approximate the value of a function for some x near c. Derivatives & Tangent-line approximation

For example, let $f(x) = e^x$. Approximate the value of $e^{0.1}$ using the tangent line at $x = 0$.



Polynomial approximation uses this logic: If linear approximation is relatively accurate for values of x close to $x = c$, maybe a quadratic approximation would be better...or maybe a cubic would be even better still... and so on. In fact, polynomial approximations usually give us a larger interval on which the approximation is relatively accurate.

Maclaurin polynomial:

If f has n derivatives at $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is the nth degree polynomial approximation of f at $c = 0$.

Example 1: Find the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Then use the function to approximate $f(0.1)$. $\boxed{c=0}$

$f(x) = e^x \rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \rightarrow f''(0) = e^0 = 1$

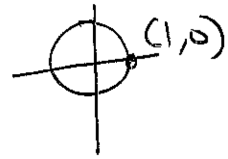
$f'''(x) = e^x \rightarrow f'''(0) = e^0 = 1$

$f^4(x) = e^x \rightarrow f^4(0) = e^0 = 1$

$P_4(x) = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$

$$\boxed{P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4}$$

$f(0.1) = 1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 + \frac{1}{24}(0.1)^4$



$c = 0$

Example 2: Find a third degree Maclaurin polynomial for $f(x) = \sin(x)$.

$f(x) = \sin x \rightarrow f(0) = \sin(0) = 0$

$f'(x) = \cos x \rightarrow f'(0) = \cos(0) = 1$

$f''(x) = -\sin x \rightarrow f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x \rightarrow f'''(0) = -\cos(0) = -1$

$$P_3(x) = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3$$

$$\boxed{P_3(x) = x - \frac{1}{6}x^3}$$

Taylor polynomial: $x = c$ $\quad c \neq 0$

TAYLOR POLYNOMIAL

If f has n derivatives at $x = c$, then the nth Taylor polynomial approximation of f at $x = c$ is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example 3: Find $P_2(x)$ for $f(x) = \sqrt{x}$ at $c = 4$.

degree = 2

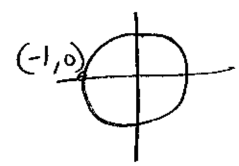
$f(x) = x^{1/2} = \sqrt{x} \rightarrow f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2} = \frac{1}{4\sqrt{x^3}} \rightarrow f''(4) = \frac{1}{4\sqrt{4^3}} = \frac{1}{32}$

$$P_2(x) = 2 + \frac{1}{4}(x-4) + \frac{1}{32}\left(\frac{1}{2!}\right)(x-4)^2$$

$$\boxed{P_2(x) = 2 + \frac{1}{4}(x-4) + \frac{1}{64}(x-4)^2}$$



Example 4: Find a second degree Taylor polynomial for $f(x) = x^2 \cos x$ centered at π. $C = \pi$

$f(x) = x^2 \cos x \longrightarrow f(\pi) = \pi^2 \cos \pi = \pi^2(-1) = -\pi^2$

$f'(x) = (2x)(\cos x) + (x^2)(-\sin x) \rightarrow f'(\pi) = 2\pi \cos \pi + \pi^2(-\sin \pi) = 2\pi(-1) + \pi^2(0) = -2\pi$

$f''(x) = (2)(\cos x) + (2x)(-\sin x) + (2x)(-\sin x) + (x^2)(-\cos x)$

$f''(\pi) = 2\cos \pi + 2\pi(\cancel{-\sin \pi}) + 2\pi(\cancel{-\sin \pi}) + \pi^2(-\cos \pi)$

$= -2 + \pi^2(1) = -2 + \pi^2$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(-2+\pi^2)}{2}(x-\pi)^2$$