Answer Key*

TAYLOR & MACLAURIN SERIES REVIEW

NO CALCULATOR

1.	The Maclaurin series for the function f is given by $f(x) = \sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$. What is the value of $f(3)$? (A) -3 (B) $-\frac{3}{7}$ (C) $\frac{4}{7}$ (D) $\frac{13}{16}$ (E) 4
2.	For $x > 0$, the power series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$ converges to which of the following? (A) $\cos x$ (B) $\sin x$ (C) $\frac{\sin x}{x}$ (D) $e^x - e^{x^2}$ (E) $1 + e^x - e^{x^2}$
3.	Let f be a function having derivatives of all orders for $x > 0$ such that $f(3) = 2$, $f'(3) = -1$, $f''(3) = 6$, and $f'''(3) = 12$. Which of the following is the third-degree Taylor polynomial for f about $x = 3$? (A) $2 - x + 6x^2 + 12x^3$ (B) $2 - x + 3x^2 + 2x^3$ (C) $2 - (x-3) + 6(x-3)^2 + 12(x-3)^3$ (D) $2 - (x-3) + 3(x-3)^2 + 4(x-3)^3$ (E) $2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$
4.	What is the value of $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$? (A) $-\frac{15}{8}$ (B) $-\frac{9}{8}$ (C) $-\frac{3}{8}$ (D) $\frac{9}{8}$ (E) $\frac{15}{8}$
5.	The third-degree Taylor polynomial for a function f about $x = 4$ is $\frac{(x-4)^3}{512} - \frac{(x-4)^2}{64} + \frac{(x-4)}{4} + 2$. What is the value of $f'''(4)$? (A) $-\frac{1}{64}$ (B) $-\frac{1}{32}$ (C) $\frac{1}{512}$ (D) $\frac{3}{256}$ (E) $\frac{81}{256}$
6.	Which of the following is the Maclaurin series for $\frac{1}{(1-x)^2}$? (A) $1 - x + x^2 - x^3 + \cdots$ (B) $1 - 2x + 3x^2 - 4x^3 + \cdots$ (C) $1 + 2x + 3x^2 + 4x^3 + \cdots$ (D) $1 + x^2 + x^4 + x^6 + \cdots$ (E) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

7. What is the coefficient of x^6 in the Taylor series for $\dfrac{e^{3x^2}}{2}$ about $x = 0$?

(A) $\dfrac{1}{1440}$ (B) $\dfrac{81}{160}$ (C) $\dfrac{9}{4}$ (D) $\dfrac{9}{2}$ (E) $\dfrac{27}{2}$

8. What is the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{e^{n+1}}$?

(A) $\dfrac{-2}{e^2-2e}$ (B) $\dfrac{-2}{e^2+2e}$ (C) $\dfrac{-2}{e+2}$ (D) $\dfrac{e}{e+2}$ (E) The series diverges.

9. Let $P(x) = 3 - 3x^2 + 6x^4$ be the fourth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

(A) 0 (B) $\dfrac{1}{4}$ (C) 6 (D) 24 (E) 144

10. Which of the following is the Maclaurin series for e^{3x}?

(A) $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

(B) $3 + 9x + \dfrac{27x^2}{2} + \dfrac{81x^3}{3!} + \dfrac{243x^4}{4!} + \cdots$

(C) $1 - 3x + \dfrac{9x^2}{2} - \dfrac{27x^3}{3!} + \dfrac{81x^4}{4!} - \cdots$

(D) $1 + 3x + \dfrac{3x^2}{2} + \dfrac{3x^3}{3!} + \dfrac{3x^4}{4!} + \cdots$

(E) $1 + 3x + \dfrac{9x^2}{2} + \dfrac{27x^3}{3!} + \dfrac{81x^4}{4!} + \cdots$

11. What is the coefficient of x^2 in the Taylor series for $\sin^2 x$ about $x = 0$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

12. The sum of the series $1 + \dfrac{2^1}{1!} + \dfrac{2^2}{2!} + \dfrac{2^3}{3!} + \cdots + \dfrac{2^n}{n!} + \cdots$ is

(A) $\ln 2$ (B) e^2 (C) $\cos 2$ (D) $\sin 2$ (E) nonexistent

13. The function f has derivatives of all orders for all real numbers with $f(0) = 3$, $f'(0) = -4$, $f''(0) = 2$, and $f'''(0) = 1$. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. What is the third-degree Taylor polynomial for g about $x = 0$?

(A) $-4x + 2x^2 + \dfrac{1}{3}x^3$

(B) $-4x + x^2 + \dfrac{1}{6}x^3$

(C) $3x - 2x^2 + \dfrac{1}{3}x^3$

(D) $3x - 2x^2 + \dfrac{2}{3}x^3$

(E) $3 - 4x + x^2 + \dfrac{1}{6}x^3$

① $f(x) = \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$ $\quad f(3) = ?$

$\downarrow$ Geometric $\quad a = \left(-\frac{x}{4}\right)^0 = 1$

$$f(x) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{4}x\right)} = \frac{1}{1+\frac{1}{4}x}$$

$$f(3) = \frac{1}{1+\frac{1}{4}(3)} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

② $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!}$

a. $\cos x = 1 - \cancel{\frac{x^2}{2!} + \frac{x^4}{4!}} - \frac{x^6}{6!}$

b. $\sin x = x - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$\boxed{\text{memorize } \cos x \text{ \& } \sin x}$

c. $\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$ ✓

$\frac{\sin x}{x}$

(3) $f(3) = 2, \quad f'(3) = -1, \quad f''(3) = 6, \quad f'''(3) = 12$

3rd-degree Taylor about $x = 3$

$$f(3) + \frac{f'(3)(x-3)^1}{1!} + \frac{f''(3)(x-3)^2}{2!} + \frac{f'''(3)(x-3)^3}{3!}$$

$$2 + \frac{(-1)(x-3)}{1} + \frac{(6)(x-3)^2}{2} + \frac{(12)(x-3)^3}{6}$$

$$= \boxed{2 - (x-3) + 3(x-3)^2 + 2(x-3)^3}$$

(4) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$ Geometric series

$$f(x) = \frac{a}{1-r} = \frac{9/5}{1-\left(\frac{-3}{5}\right)} = \frac{\frac{9}{5}}{\frac{8}{5}} = \frac{9}{5} \cdot \frac{5}{8}$$

$$= \boxed{\frac{9}{8}}$$

$$a = \frac{(-3)^{1+1}}{5^1} = \frac{(-3)^2}{5}$$

$$= \frac{9}{5}$$

(5) 3^{rd}-degree about $x = 4$

$$\frac{(x-4)^3}{512} - \frac{(x-4)^2}{64} + \frac{(x-4)}{4} + 2 \qquad f'''(4) = ?$$

$$\uparrow$$

$$\frac{f'''(4)(x-4)^3}{3!} = \frac{f'''(4)(x-4)^3}{6}$$

$$\frac{x}{6} \times \frac{1}{512}$$

$$512x = 6$$

$$x = \frac{6}{512} = \boxed{\frac{3}{256}}$$

(6) Maclaurin series for $\frac{1}{(1-x)^2}$

about $x=0$

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$$

$$1 + 2x + \frac{6x^2}{2!} + \frac{24x^3}{3!} + \ldots$$

$$\boxed{1 + 2x + 3x^2 + 4x^3 + \ldots}$$

$f(x) = (1-x)^{-2} \rightarrow f(0) = (1-0)^{-2} = 1$

$f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} \rightarrow f'(0) = 2(1-0)^{-3} = 2$

$f''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4} \rightarrow f''(0) = 6(1-0)^{-4} = 6$

$f'''(x) = -24(1-x)^{-5}(-1) = 24(1-x)^{-5} \rightarrow f'''(0) = 24(1-0)^{-5} = 24$

7 coeff of x^6 for Taylor series for $\frac{e^{3x^2}}{2}$ about $x=0$

$$\frac{f^6(0)\,x^6}{6!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!}$$

$$\frac{e^{3x^2}}{2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{2 \cdot n!}$$

n=6: $\cancel{\frac{(3x^2)^6}{2 \cdot 6!} = \frac{729x^{12}}{1440}}$

n=3: $\frac{(3x^2)^3}{2 \cdot 3!} = \frac{27x^6}{12} = \boxed{\frac{9}{4}x^6}$

⑧ sum $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}}$

Geometric Series

Sum= $\frac{a}{1-r} = \frac{-2/e^2}{1-\left(\frac{-2}{e}\right)}$

$a = \frac{(-2)^1}{e^{1+1}} = \frac{-2}{e^2}$

$= \frac{\frac{-2}{e^2}}{1+\frac{2}{e}} = \frac{\frac{-2}{e^2}}{\frac{e+2}{e}}$

$\frac{e}{e} + \frac{2}{e} = \frac{e+2}{e}$

$\frac{-2}{e^{\cancel{2}}} \cdot \frac{\cancel{e}}{e+2}$

$= \frac{-2}{e(e+2)} = \boxed{\frac{-2}{e^2+2e}}$

⑨ $P(x) = 3 - 3x^2 + 6x^4$ about $x=0$

$f^{(4)}(0) = ?$

$f(0) + \frac{f''(0)x^2}{2!} + \frac{f^4(0)x^4}{4!}$

$3 - \frac{\# x^2}{2!} + \frac{\# x^4}{4!}$

n=0 n=1 n=2

general term= $\frac{(-1)^n x^{2n} f^{2n}(0)}{(2n)!}$

$\underline{n=2}$ $\frac{(-1)^2 x^4 f^4(0)}{4!}$

$= \frac{x^4 f^4(0)}{24} \longrightarrow$

$\frac{\#}{24} = 6$

$\# = 24(6)$

$= \boxed{144}$

(10) e^{3x} about $x=0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$\underline{n=0}$: $\frac{(3x)^0}{0!} = 1$

$\underline{n=1}$; $\frac{(3x)^1}{1!} = 3x$

$$\boxed{1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \dots}$$

$\underline{n=2}$: $\frac{(3x)^2}{2!} = \frac{9x^2}{2}$

$\underline{n=3}$: $\frac{(3x)^3}{3!} = \frac{27x^3}{3!}$

(11) coeff of x^2 about $\sin^2 x$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$(\sin x)^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$\underline{n=1}$: $\cancel{\frac{(-1)^1 (x^2)^{2(1)+1}}{(2n+1)!}}$

$\cancel{(-1) x^6}$

$\underline{n=0}$: $\frac{\cancel{(-1)^0}(x^2)^{2(0)+1}}{(2(0)+1)!} = \frac{x^2}{1!}$

$= x^2$

$= \boxed{1}$

(12) $1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots + \frac{2^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} \qquad \boxed{e^2}$$

(13) $f(0) = 3, \; f'(0) = -4, \; f''(0) = 2, \; f'''(0) = 1$

$$g(x) = \int_0^x f(t)\,dt$$

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= 3 + -4x + \frac{2x^2}{2} + \frac{1x^3}{6}$$

$$f(x) = 3 - 4x + x^2 + \tfrac{1}{6}x^3$$

$$g(x) = \int 3 - 4x + x^2 + \tfrac{1}{6}x^3$$

$$3x - \frac{4x^2}{2} + \frac{x^3}{3} + \frac{\frac{1}{6}x^4}{4}$$

$$= \boxed{3x - 2x^2 + \tfrac{1}{3}x^3 +} \tfrac{1}{24}x^4$$