AP Calculus BC **Name: ______________________________**
Midterm Review

1. What are all of the horizontal asymptotes of all the solutions of the logistics differential equation $\frac{dy}{dx} = y(16 - 2y)$?

2. $\int sec^5 x tan^3 x dx =$

3. Evaluate the integral: $\int e^x cosx dx$

4. What is the carrying capacity for a population whose growth rate is modeled by $\frac{dP}{dt} = 45P - 9P^2$?

5. Evaluate the integral: $\int \frac{5}{x^2+8x+18} dx$

6. Evaluate the integral: $\int sin^3(x) \, cos^2(x) dx$

7. $\int -5xcos2xdx$

8. $\int \frac{4x-1}{x^2-3x-40} dx$

9. $\int xcos(2\pi x^2)dx$

10. Evaluate the integral: $\int x\sqrt{x+1}dx$

11. Evaluate the integral: $\int 3x(x^2-1)^4 dx$

12. $\int \frac{5}{\sqrt{1-16x^2}} dx$

13. Which of the following integrals are divergent?

I. $\int_2^\infty \frac{x}{(1+x^2)^2 dx}$ II. $\int_1^\infty \frac{1}{x} dx$ III. $\int_2^\infty cos2x \, dx$

14. What is the value of $\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n$?

15. Which of the following series converges?

I. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ II. $\sum_{n=1}^{\infty}\frac{3^n}{n!}$ III. $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$

16. Determine whether the following sequence converges or diverges. If it converges, find its limit.

$\left\{\frac{(n-2)!}{(n+1)!}\right\}$, $n = 0, 1, 2, \ldots$

17. Investigate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ for convergence or divergence.

18. Find the third term of the sequence $\left\{\frac{(-1)^n(2^n+1)}{n!}\right\}$, $n = 1, 2, 3, \ldots$

19. Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent or divergent. If convergent, classify the series as absolutely convergent or conditionally convergent.

20. Find the number of terms necessary to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ with an error of less than or equal to 0.001.

21. Determine if the following sequence converges or diverges. If it converges, find its limit.

$\left\{\frac{5n-1}{3n+1}\right\}$, $n = 1, 2, 3, \ldots$

22. Determine which series diverges.

a) $\sum_{n=0}^{\infty}\frac{n!}{6n!-1}$ b) $\sum_{n=1}^{\infty}\frac{1}{n^6}$ c) $\sum_{n=0}^{\infty}5\left(\frac{1}{10}\right)^n$ d) $\sum_{n=0}^{\infty}\frac{n}{2^n}$

23. Determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{3}{(2n-1)(2n+1)}$.

24. Determine if the series converges or diverges. $\sum_{n=1}^{\infty}\left(\frac{2n-1}{3n+5}\right)^n$

25. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{n}{n+5}$ II. $\sum_{n=1}^{\infty}\frac{1}{n-3}$ III. $\sum_{n=1}^{\infty}\frac{1}{n}$

26. What is the radius of convergence for the power series $\sum_{n=0}^{\infty}\frac{(x-5)^n}{2\cdot 3^{n+1}}$?

27. Find the interval of convergence for a power series that is centered at -2 for the function $f(x)=\frac{3}{2-4x}$.

28. Let f be the function given by $f(x)=\ln(3-x)$. The third-degree Taylor polynomial for f about $x=2$ is

29. Write out the first four terms of the Taylor series for f(x) = xcosx about x = 0.

30. The third-degree Taylor polynomial for a function f about x = 4 is $\frac{(x-4)^3}{512}-\frac{(x-4)^2}{64}+\frac{(x-4)}{4}+2$. What is the value of $f''(4)$?

31. Let f be a function with $f(3)=2,\ f'(3)=-1,\ f''(3)=6,$ and $f'''(3)=12.$ Which of the following is the third-degree Taylor polynomial for f about $x=3$?

32. Find the interval of convergence of the Maclaurin series for $f(x)=e^{-2x}$.

33. Use the ~~4th~~ 5th degree Taylor Series for sinx about x = 0 to determine whether f has a relative minimum, relative maximum, or neither at x = 0.

34. If $f(x)=e^{-x^2}$. Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about x = 0,

① $\frac{dy}{dx} = y\left(\frac{16}{2} - \frac{2y}{2}\right)$

$2y(8-y)$ $\boxed{\text{H.A. } y=0 \text{ \& } y=8}$

② $\int \sec^5 x \tan^3 x \, dx$ (keep out secxtanx then change everything to secx.)

$\int \sec^4 x \tan^2 x (\sec x \tan x) \, dx$

$\int (\sec^4 x)(\sec^2 x - 1)(\sec x \tan x) dx$

$\int (\sec^6 x - \sec^4 x)(\sec x \tan x) \, dx$

$u = \sec x$
$du = \sec x \tan x \, dx$

$\int u^6 - u^4 \, du$

$\frac{1}{7}u^7 - \frac{1}{5}u^5 + C$

$\boxed{\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C}$

③ $\int e^x \cos x\,dx$

$u = \cos x \qquad v = e^x$

$du = -\sin x\,dx \qquad dv = e^x dx$

$(\cos x)(e^x) - \int (-\sin x)(e^x)dx$

$e^x \cos x + \underbrace{\int e^x \sin x\,dx}$

$u = \sin x \qquad v = e^x$

$du = \cos x\,dx \qquad dv = e^x dx$

$(\sin x)(e^x) - \int (\cos x)(e^x dx)$

$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$

$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$

$\int e^x \cos x\,dx = \boxed{\frac{1}{2}(e^x \cos x + e^x \sin x) + C}$

④ $\frac{dP}{dt} = 45P - 9P^2$

$= 9P(5-P)$

$\boxed{\text{carrying capacity} = 5}$

⑤ $\int \frac{5}{x^2+8x+18}\,dx$

$x^2+8x+\left(\frac{8}{2}\right)^2+18-\left(\frac{8}{2}\right)^2$

$(x+4)^2+2$

$5\int \frac{1}{(x+4)^2+2}\,dx$

$u=x+4$
$du=dx$
$a=\sqrt{2}$

$5\int \frac{du}{u^2+a^2}$

$5\left(\frac{1}{\sqrt{2}}\right)\arctan\left(\frac{x+4}{\sqrt{2}}\right)+C = \boxed{\frac{5}{\sqrt{2}}\arctan\frac{x+4}{\sqrt{2}}+C}$

⑥ $\int \sin^{3}x\cos^2x\,dx$ (odd)

(Keep out sinx and change everything to cosx)

$\int \underbrace{\sin^2 x}\cos^2x\,(\sin x)\,dx$

$\int (1-\cos^2x)(\cos^2x)(\sin x\,dx)$

$-1\int (\cos^2x-\cos^4x)(\sin x\,dx)$

$u=\cos x$
$du=-\sin x\,dx$

$-1\int u^2-u^4\,du$

$-1\left[\frac{1}{3}u^3-\frac{1}{5}u^5\right]+C$

$\boxed{-\frac{1}{3}\cos^3x+\frac{1}{5}\cos^5x+C}$

(7) $\int -5x\cos 2x\,dx$

$u = -5x \qquad v = \frac{1}{2}\sin 2x$

$du = -5\,dx \qquad dv = \cos 2x\,dx$

$(-5x)(\frac{1}{2}\sin 2x) - \int (\frac{1}{2}\sin 2x)(-5)\,dx$

$+ \frac{5}{2}\int \sin 2x\,dx$

$+ \frac{5}{2}\left[-\frac{1}{2}\cos 2x\right]$

$$\boxed{-\frac{5}{2}x\sin 2x - \frac{5}{4}\cos 2x + C}$$

(8) $\int \frac{4x-1}{x^2-3x-40}\,dx$

$x^2 - 3x - 40 = (x-8)(x+5)$

$\frac{4x-1}{(x-8)(x+5)} = \frac{A}{x-8} + \frac{B}{x+5}$

$4x-1 = A(x+5) + B(x-8)$

$\underline{x=-5}$: $4(-5)-1 = B(-5-8)$

$-21 = -13B$

$B = 21/13$

$\underline{x=8}$: $4(8)-1 = A(8+5)$

$31 = 13A$

$A = 31/13$

$\int \frac{31/13}{x-8}\,dx + \int \frac{21/13}{x+5}\,dx$

$\frac{31}{13}\int \frac{1}{x-8}\,dx + \frac{21}{13}\int \frac{1}{x+5}\,dx$

$u = x-8 \qquad u = x+5$

$du = dx \qquad du = dx$

$\frac{31}{13}\int \frac{du}{u} \qquad \frac{21}{13}\int \frac{du}{u}$

$$\boxed{\frac{31}{13}\ln|x-8| + \frac{21}{13}\ln|x+5| + C}$$

9) $\frac{1}{4\pi}\int 4\pi\, x\cos(2\pi x^2)\,dx$

$u = 2\pi x^2$

$du = 4\pi x\,dx$

$\frac{1}{4\pi}\int \cos u\,du$

$\frac{1}{4\pi}\sin u + C$

$$\boxed{\frac{1}{4\pi}\sin(2\pi x^2) + C}$$

10) $\int x\sqrt{x+1}\,dx$

$u = x+1$

$du = dx$

$u-1 = x$

$\int (u-1)\,u^{1/2}\,du$

$\int u^{3/2} - u^{1/2}\,du$

$\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$$\boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}$$

11. $\frac{1}{2}3\int \cancel{3}^{2}x(x^2-1)^4\,dx$

$u=x^2-1$
$du=2x\,dx$

$\frac{3}{2}\int u^4\,du$

$\frac{3}{2}\left[\frac{1}{5}u^5\right]+C$

$\boxed{\frac{3}{10}(x^2-1)^5+C}$

12. $\frac{1}{4}5\int \frac{\cancel{5}\,4}{\sqrt{1-16x^2}}\,dx = \frac{1}{4}\cdot 5\int \frac{du}{\sqrt{a^2-u^2}}$

$u=4x$
$du=4dx$
$a=1$

$\boxed{\frac{5}{4}\arcsin 4x+C}$

(13) I. $\int_2^\infty \frac{x}{(1+x^2)^2}dx$ $\quad \lim_{b\to\infty} \frac{1}{2}\int_2^b \frac{2x}{(1+x^2)^2}dx$ $\quad u=1+x^2$

$du=2x\,dx$

$$\frac{1}{2}\int \frac{du}{u^2}$$

$$\frac{1}{2}\int u^{-2}du$$

$$\frac{1}{2}\left[-u^{-1}\right]$$

$$-\frac{1}{2}(1+x^2)^{-1} = \left.\frac{-1}{2(1+x^2)}\right]_2^b$$

$$\lim_{b\to\infty}\left(\frac{-1}{2(1+b^2)} + \frac{1}{2(1+2^2)}\right) = 0 + \frac{1}{10} = \frac{1}{10}$$

converges

* II. $\int_1^\infty \frac{1}{x}$ "Special rule"

$p=1$ since $p\leq 1$, diverges

* III. $\int_2^\infty \cos 2x \to \lim_{b\to\infty}\int_2^b \cos 2x\,dx$

$$\left.\frac{1}{2}\sin 2x\right]_2^b$$

$$\lim_{b\to\infty}\left(\frac{1}{2}\sin bx - \frac{1}{2}\sin 4\right)$$

DNE

diverges

(14) $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$ Geometric $\to$ $S = \frac{a}{1-r}$

$r = -1/3$

converges

$$S = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \boxed{\frac{3}{4}}$$

(15) I. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series test $p = 1/2$ diverges

II. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{\cancel{(n+1)!}_{(n+1)(n!)}} \cdot \frac{\cancel{n!}}{\cancel{3^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{3}{n+1} \right| = \frac{3}{\infty} = 0 < 1 \quad \text{converges}$$

III. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ geometric series, $r = \frac{e}{\pi} < 1$

converges

(16) sequence $\rightarrow \left\{\frac{(n-2)!}{(n+1)!}\right\}$

$$\lim_{n\to\infty} \frac{\cancel{(n-2)!}}{\cancel{(n+1)!}} = \lim_{n\to\infty} \frac{1}{(n+1)(n)(n-1)} = \frac{1}{\infty} = 0$$

$$(n+1)(n)(n-1)\cancel{(n-2)!}$$

$\boxed{\text{converges to } 0}$

(17) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Alt. series Test

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\frac{1}{n+1} \leq \frac{1}{n}$ ✓

$\boxed{\text{converges}}$

(18) $\left\{\frac{(-1)^n(2^n+1)}{n!}\right\}$

$$\frac{(-1)^3(2^3+1)}{3!} = \frac{-1(9)}{6} = \boxed{\frac{-3}{2}}$$

(19) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ ① Alt. series Test $\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$ ✓ $\frac{1}{(n+1)^{1/2}} \leq \frac{1}{n^{1/2}}$ ✓

converges

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series test

$p = 1/2$ diverges

$\boxed{\text{Conditionally Convergent}}$

(20) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

$\frac{1}{n+1+1} = \frac{1}{n+2} \leq \frac{1}{1000}$

$n+2 \leq 1000$

$n \leq \boxed{998}$

(21) sequence $\rightarrow \left\{ \frac{5n-1}{3n+1} \right\}$

$\lim_{n\to\infty} \frac{5n-1}{3n+1} = \frac{5}{3}$ $\boxed{\text{converges to } 5/3}$

(22) a) $\sum_{n=0}^{\infty} \frac{n!}{6n!-1}$ n^{th} term test $\lim_{n\to\infty} \frac{n!}{6n!-1} = \frac{1}{6} \neq 0$ diverges

b) $\sum_{n=1}^{\infty} \frac{1}{n^6}$ p-series test converges

$p=6$

c) $\sum_{n=0}^{\infty} 5\left(\frac{1}{10}\right)^n$ Geom. series+ converges

$r = 1/10$

d) $\sum_{n=0}^{\infty} \frac{n}{2^n}$ Ratio Test

$\lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2n} \right| = \frac{1}{2} < 1$

converges

(23) $\sum_{n=1}^{\infty} \frac{3}{(2n-1)(2n+1)}$ Telescoping series $\rightarrow$ Converges

(24) $\sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+5}\right)^n$ Root Test

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n-1}{3n+5}\right)^n} = \lim_{n\to\infty} \frac{2n-1}{3n+5} = \frac{2}{3} < 1$

Converges

(25) I. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ nth root test

$\lim_{n\to\infty} \frac{n}{n+5} = 1 \neq 0$ diverges

II. $\sum_{n=1}^{\infty} \frac{1}{n-3}$ $a_n = \frac{1}{n-3}$ $b_n = \frac{1}{n}$

bigger

$\sum_{n=1}^{\infty} \frac{1}{n}$ p-series test, $p=1$ diverges

III. $\sum_{n=1}^{\infty} \frac{1}{n}$ p-series test, $p=1$ diverges

(26) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{2\cdot 3^{n+1}}$ center $=5$

$$\lim_{n\to\infty}\left|\frac{(x-5)^{\cancel{n+1}}}{\cancel{2}\cdot 3^{\cancel{n+2}}}\cdot\frac{\cancel{2\cdot 3^{n+1}}}{\cancel{(x-5)^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)}{3}\right| = \left|\frac{x-5}{3}\right|$$

$$\left|\frac{x-5}{3}\right| < 1$$

$|x-5| < 3$ $\boxed{\text{Radius} = 3}$

(27) $\dfrac{3}{2-4x}$ center $=-2$

$$\frac{3}{2-4(x+2)+8} = \frac{3}{10-4(x+2)} = \frac{3/10}{1-\frac{4(x+2)}{10}}$$

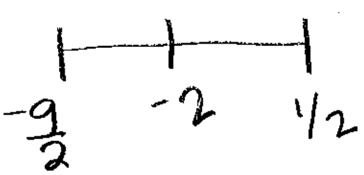
$a = 3/10$

$r = \dfrac{4(x+2)}{10} = \dfrac{2(x+2)}{5}$

$$\left|\frac{2(x+2)}{5}\right| < 1$$

$|x+2| < \frac{5}{2}$ $R = 5/2$

$\boxed{\left(-\frac{9}{2}, \frac{1}{2}\right)}$



$$-2+\frac{5}{2} = -\frac{4}{2}+\frac{5}{2} = \frac{1}{2}$$

$$-2-\frac{5}{2} = \frac{-4}{2}-\frac{5}{2} = \frac{-9}{2}$$

28. $f(x) = \ln(3-x)$ about $x=2$

$f(x) = \ln(3-x) \longrightarrow \ln(3-2) = \ln 1 = 0$

$f'(x) = \frac{-1}{3-x} = -1(3-x)^{-1} \rightarrow \frac{-1}{3-2} = -1$

$f''(x) = (3-x)^{-2} \rightarrow \frac{1}{(3-2)^2} = 1$

$f'''(x) = -2(3-x)^{-3} \rightarrow \frac{-2}{(3-2)^3} = -2$

$0 + -1(x-2) + \frac{1(x-2)^2}{2!} - \frac{2(x-2)^3}{3!}$ $\qquad \frac{-2}{6} = \frac{-1}{3}$

$$\boxed{-(x-2) + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3}}$$

29. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$n=0$: $\frac{(-1)^0 x^0}{0!} = 1$

$n=1$: $\frac{(-1)^1 x^2}{2!} = \frac{-x^2}{2!}$

$n=2$: $\frac{(-1)^2 x^4}{4!} = \frac{x^4}{4!}$

$n=3$: $\frac{(-1)^3 x^6}{6!} = \frac{-x^6}{6!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$x\cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$

$$= \boxed{x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}}$$

(30) $f(x) = \frac{1}{512}(x-4)^3 - \frac{1}{64}(x-4)^2 + \frac{1}{4}(x-4) + 2$

$f'(x) = \frac{3}{512}(x-4)^2 - \frac{2}{64}(x-4) + \frac{1}{4} + \cancel{0}$

$f''(x) = \frac{6}{512}(x-4) - \frac{2}{64} + \cancel{0} + \cancel{0}$

$f''(4) = \cancel{\frac{6}{512}(4-4)} - \frac{2}{64} = \frac{-2}{64} = \boxed{-\frac{1}{32}}$

(31) $f(3) + f'(3)(x-3) + \frac{f''(3)(x-3)^2}{2!} + \frac{f'''(3)(x-3)^3}{3!}$

$2 - 1(x-3) + \frac{6(x-3)^2}{2!} + \frac{12(x-3)^3}{3!}$

$\boxed{2 - (x-3) + 3(x-3)^2 + 2(x-3)^3}$

(32) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \rightarrow e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^n x^n}{n!}$

$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\, 2^{\cancel{n+1}} x^{\cancel{n+1}}}{\cancel{(n+1)!}\ (n+1)(\cancel{n!})} \cdot \frac{\cancel{n!}}{\cancel{(-1)^n 2^n x^n}} \right| = \lim_{n\to\infty} \left| \frac{2x}{n+1} \right|$

$|2x| \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 < 1$ converges by Ratio test

$R = \infty$

$\boxed{(-\infty, \infty)}$

(33) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\underline{n=0}$: $\frac{(-1)^0 x^1}{1!} = x$

$\underline{n=1}$: $\frac{(-1)^1 x^3}{3!} = \frac{-1}{6}x^3$

$\underline{n=2}$: $\frac{(-1)^2 x^5}{5!} = \frac{1}{120}x^5$

$f(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$f'(x) = 1 - x^2 + \frac{5}{120}x$

$f'(0) = 1$

$\uparrow$

$\boxed{\text{neither}}$, $x=0$ is not a critical point

(34) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum \frac{(-1)^n x^{2n}}{n!}$

$\underline{n=0}$: $\frac{(-1)^0 x^0}{0!} = 1$

$\underline{n=1}$: $\frac{(-1)^1 x^2}{1!} = -x^2$

$\underline{n=2}$: $\frac{(-1)^2 x^4}{2!} = \frac{1}{24}x^4$

$\underline{n=3}$: $\frac{(-1)^3 x^6}{3!} = -\frac{1}{6}x^5$

$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{720}x^6$

$\int_0^x e^{-t^2} = \int_0^x 1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 dt$

$t - \frac{1}{3}t^3 + \frac{1}{10}t^5 - \frac{1}{42}t^7 \Big]_0^x$

$\boxed{x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7}$