LIMITS REVIEW

TECHNIQUES FOR EVALUATING LIMITS: Approach in this order.

1. Direct substitution
2. Factor to rename the function
3. Rationalize the numerator
4. Use special trig limits

   $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

5. Use L'Hopital's Rule if the function can be put in an indeterminate form. $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

   $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

ONE-SIDED LIMITS AND CONTINUITY

Definition of continuity at a point: $f(x)$ is continuous at $x = c$ if and only if theses 3 conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists. $\left(\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)\right)$
3. $\lim_{x \to c} f(x) = f(c)$.

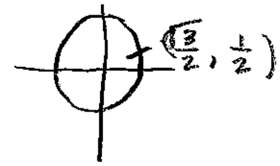
continuous

INTERMEDIATE VALUE THEOREM

If $f(x)$ is continuous on $[a, b]$ and k is between $f(a)$ and $f(b)$, then there exists at least one value c in the interval (a, b) such that $f(c) = k$.

- continuous
- k is b/t f(a) & f(b)

} f(c) = k by Intermediate Value Thm



Limits Review

Direct substitution

1. $\lim_{x \to \frac{\pi}{6}} \sec^2 x =$ $\quad (\sec(\frac{\pi}{6}))^2 = (\frac{2}{\sqrt{3}})^2 = \frac{4}{3}$

(A) $\frac{3}{4}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{4}{3}$ (D) $\frac{2\sqrt{3}}{3}$

2. If $f(x) = \begin{cases} x^2+3, & x \neq 1 \\ 1, & x = 1 \end{cases}$, then $\lim_{x \to 1} f(x) =$

LOOK @ Left & Right sided limits

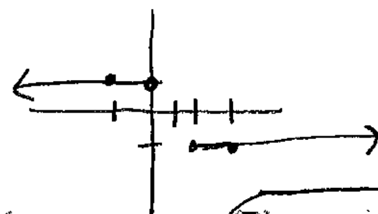
$\lim_{x \to 1^-} f(x) = 1^2+3 = 4 \qquad \lim_{x \to 1^+} f(x) = 1^2+3 = 4$

(A) 1 (B) 2 (C) 3 (D) 4

3. $\lim_{x \to 1} \frac{|x-1|}{1-x} =$

X	Y
1	undefined
2	-1
3	-1
0	1
-1	1

Graph!



$\left.\begin{array}{l} \lim_{x \to 1^-} = 1 \\ \lim_{x \to 1^+} = -1 \end{array}\right\}$ not equal

(A) -2 (B) -1 (C) 1 (D) nonexistent

4. Let f be a function given by $f(x) = \begin{cases} 3-x^2, & \text{if } x < 0 \\ 2-x, & \text{if } 0 \leq x < 2 \\ \sqrt{x-2}, & \text{if } x > 2 \end{cases}$.

Which of the following statements are true about f?

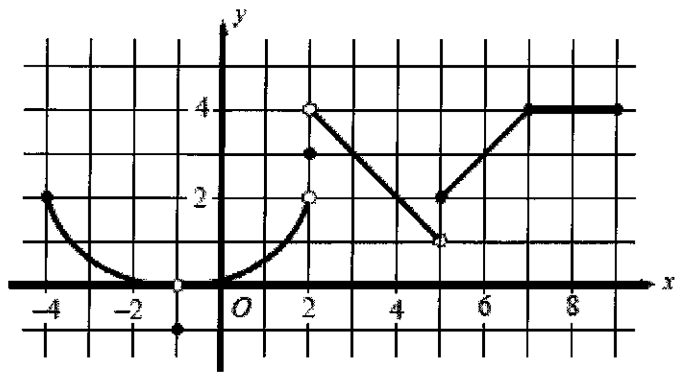
~~I.~~ $\lim_{x \to 0} f(x) = 2$ $\quad \lim_{x \to 0^-} = 3-(0)^2 = 3 \quad \lim_{x \to 0^+} = 2-0 = 2 \quad \lim_{x \to 0}$ DNE

True! II. $\lim_{x \to 2} f(x) = 0$ $\quad \lim_{x \to 2^-} = 2-2 = 0 \quad \lim_{x \to 2^+} \sqrt{2-2} = 0 \rightarrow \lim_{x \to 2} = 0$

~~III.~~ $\lim_{x \to 1} f(x) = \lim_{x \to 6} f(x)$ $\quad \lim_{x \to 1} = 2-1 = 1 \quad \lim_{x \to 6} = \sqrt{6-2} = 2 \quad \neq$

(A) I only (B) II only (C) II and III only (D) I, II, and III

Questions 5-11 refer to the following graph.



The figure above shows the graph of $y = f(x)$ on the closed interval $[-4, 9]$.

5. Find $\lim_{x \to -1} \cos(f(x))$. $\cos(0) = \boxed{1}$

6. Find $\lim_{x \to 2^-} f(x)$. Left $= \boxed{2}$

7. Find $\lim_{x \to 2^+} f(x)$. Right $= \boxed{4}$

8. Find $\lim_{x \to 2} f(x)$. $\boxed{\text{DNE}}$ b/c $\lim_{x \to 2^-} \neq \lim_{x \to 2^+}$

9. Find $f(2)$. $= \boxed{3}$

10. Find $\lim_{x \to 5^-} \arctan(f(x))$. $\tan^{-1}(1) = \boxed{\frac{\pi}{4}}$

11. Find $\lim_{x \to 5^+} [x f(x)]$. $5(2) = \boxed{10}$

12. $\lim_{x \to \pi/3} \dfrac{\sin(\frac{\pi}{3} - x)}{\frac{\pi}{3} - x} =$ $\dfrac{\sin(\frac{\pi}{3} - \frac{\pi}{3})}{\frac{\pi}{3} - \frac{\pi}{3}} = \dfrac{0}{0}$

use L'Hopital's Rule

$\lim_{x \to \pi/3} \dfrac{\cos(\frac{\pi}{3} - x)(-1)}{-1} = \dfrac{\cos(\frac{\pi}{3} - \frac{\pi}{3})(-1)}{(-1)}$

(A) -1 (B) 0 (C) $\frac{\sqrt{3}}{2}$ (D) 1

$= \dfrac{(1)(-1)}{(-1)} = 1$

13. $\lim_{x \to 0} \dfrac{\sin 3x}{\sin 2x} =$ $\dfrac{\sin 0}{\sin 0} = \dfrac{0}{0}$ use L'Hopital's Rule $\lim_{x \to 0} \dfrac{3\cos 3x}{2\cos 2x} = \dfrac{3\cos(0)}{2\cos(0)} = \dfrac{3}{2}$

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{3}{2}$ (D) nonexistent

14. $\lim_{x \to 0} \dfrac{(4+x)^{\frac{1}{2}} - 2}{x}$ $\lim_{x \to 0} \dfrac{\sqrt{4+x} - 2}{x} =$ $\dfrac{\sqrt{4+0} - 2}{0} = \dfrac{0}{0}$ use L'Hopital Rule $\to$ $\lim_{x \to 0} \dfrac{\frac{1}{2}(4+x)^{-1/2}}{1}$

$\frac{1}{2}(4+0)^{-1/2} = \dfrac{\frac{1}{2}}{\sqrt{4}}$

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) nonexistent

$= \dfrac{\frac{1}{2}}{2} = \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}$

$\lim_{x \to 1} \frac{(3+x)^{1/2}-2}{x^3}$ ↑

L'Hopitals Rule $\lim_{x \to 1} \frac{\frac{1}{2}(3+x)^{-1/2}}{3x^2} = \frac{\frac{1}{2}(3+1)^{-1/2}}{3(1)^2}$

15. $\lim_{x \to 1} \frac{\sqrt{3+x}-2}{x^3-1} =$ $\frac{\sqrt{3+1}-2}{(1)^3-1} = \frac{0}{0}$

$= \frac{\frac{1}{2}}{3\sqrt{4}} = \frac{\frac{1}{2}}{6}$

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\sqrt{3}$ (D) nonexistent

$\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

L'Hopitals Rule

16. $\lim_{\theta \to 0} \frac{\theta + \theta\cos\theta}{\sin\theta\cos\theta} =$ $\frac{0 + 0\cos 0}{\sin 0 \cos 0} = \frac{0}{0}$

$\lim_{\theta \to 0} \frac{1 + \cos\theta + \theta(-\sin\theta)}{\cos\theta\cos\theta + \sin\theta(-\sin\theta)} = \frac{1 + \cos 0 + \cancel{0(-\sin 0)}}{\cos 0 \cos 0 + \sin 0(-\sin 0)}$

$= \frac{1+1}{(1)(1)+(0)(0)}$

$= \frac{2}{1} = 2$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

L'Hopital's Rule

17. $\lim_{x \to 0} \frac{\tan 3x}{x} =$ $\frac{\tan(0)}{0} = \frac{0}{0}$

$\lim_{x \to 0} \frac{3\sec^2 3x}{1}$

$= 3\sec^2(0)$

$= 3\left(\frac{1}{1}\right) = 3$

(A) 0 (B) $\frac{1}{3}$ (C) 1 (D) 3

18. $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} =$

$\frac{1}{x}\frac{(3)}{(3)} - \frac{1}{3}\frac{(x)}{(x)}$

$\frac{\frac{3-x}{3x}}{x-3} = \frac{\cancel{3-x}}{3x} \cdot \frac{1}{\cancel{x-3}} = \frac{-1}{3x}$ $\quad -(x-3)$

$\lim_{x \to 3} \frac{-1}{3x} = -\frac{1}{9}$

(A) $-\frac{1}{9}$ (B) $\frac{1}{9}$ (C) -9 (D) 9

$(2+ax)^{1/2}$

19. If $\lim_{x \to 0} \frac{\sqrt{2+ax}-\sqrt{2}}{x} = \sqrt{2}$ what is the value of a?

$\lim_{x \to 0} \frac{\frac{1}{2}(2+ax)^{-1/2}(a)}{1}$

$\frac{1}{2}(2+0)^{-1/2}(a) = \sqrt{2}$

$\frac{a}{2\sqrt{2}} = \sqrt{2}$

$a = \sqrt{2}(2\sqrt{2})$

$a = 2(2) \rightarrow \boxed{a=4}$

Limit Definition of a Derivative at $x=0$

20. Find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, if $f(x) = \sqrt{2x+1}$.

$f(x) = (2x+1)^{1/2}$

$f'(x) = \frac{1}{2}(2x+1)^{-1/2}(2)$

$(2x+1)^{-1/2}$

plug in $x=0$

$(2(0)+1)^{-1/2} = 1^{-1/2} = \boxed{1}$

21. Find $\lim_{x \to 0} \frac{f(x)-g(x)}{\sqrt{g(x)+7}}$, if $\lim_{x \to 0} f(x) = 2$ and $\lim_{x \to 0} g(x) = -3$.

$\frac{2-(-3)}{\sqrt{(-3)+7}} = \frac{5}{\sqrt{4}} = \boxed{\frac{5}{2}}$

22. Find $\lim_{x\to\sqrt{3}} g(x)$, if $\lim_{x\to\sqrt{3}} \frac{1}{x^2+g(x)}=\frac{1}{5}$.

$\frac{1}{(\sqrt{3})^2+g(x)} = \frac{1}{5} \rightarrow \frac{1}{3+g(x)} = \frac{1}{5}$

$\lim_{x\to\sqrt{3}}(2)=\boxed{2} \leftarrow g(x)=2$

23. Let f be a function defined by $f(x)=\begin{cases}\frac{x^2-a^2}{x-a}, & \text{if } x\neq a\\ 4, & \text{if } x=a\end{cases}$. If f is continuous for all real numbers x, what is the value of a?

$\boxed{\lim_{x\to a^-} = \lim_{x\to a^+} = f(a)}$

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) 2

$\lim_{x\to a} \left(\frac{x^2-a^2}{x-a}\right)$

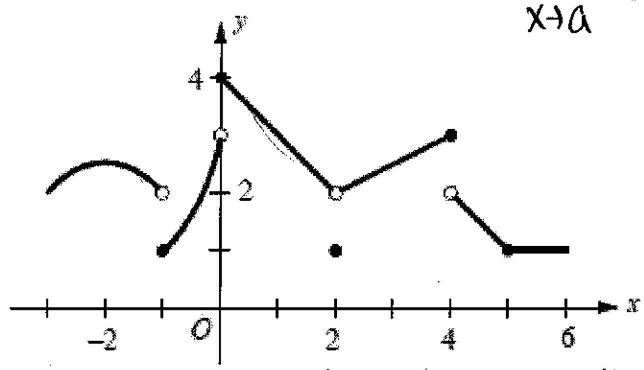
$\lim_{x\to a} \frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}} = a+a$

$= 2a$

$2a=4$

$a=2$

24.



$\lim_{x\to a^+} = \lim_{x\to a^-}$ $\lim_{x\to a} f(x) \neq f(a)$

The graph of a function f is shown above. If $\lim_{x\to a} f(x)$ exists and f is not continuous at then $a =$

Limit DNE | Limit DNE | $\lim_{x\to 2} = 2$ | Limit DNE

~~(A)~~ −1 ~~(B)~~ 0 (C) 2 $f(2)=1$ ~~(D)~~ 4

25. If $f(x)=\begin{cases}\frac{\sqrt{3x-1}-\sqrt{2x}}{x-1}, & \text{for } x\neq 1\\ a, & \text{for } x=1\end{cases}$, and if f is continuous at $x=1$ then $a =$

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{2}}{4}$ (C) $\sqrt{2}$ (D) 2

$\lim_{x\to 1} \frac{\sqrt{3x-1}-\sqrt{2x}}{x-1}\left(\cdot\frac{\sqrt{3x-1}+\sqrt{2x}}{\sqrt{3x-1}+\sqrt{2x}}\right) = \lim_{x\to 1}\left(\frac{3x-1-2x}{(x-1)(\sqrt{3x-1}+\sqrt{2x})}\right) = \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{3x-1}+\sqrt{2x})}$

$\lim_{x\to 1}\frac{1}{\sqrt{3x-1}+\sqrt{2x}} = \frac{1}{\sqrt{2}+\sqrt{2}}$

$f(1)=\frac{1}{2\sqrt{2}}$

$a=\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}} \leftarrow = \frac{1}{2\sqrt{2}}$

26. Let f be a continuous function on the closed interval $[-2, 7]$. If $f(-2) = 5$ and $f(7) = -3$, then the Intermediate Value Theorem guarantees that

(A) ~~$f'(c) = 0$ for at least one c between -2 and 7~~

(B) ~~$f'(c) = 0$ for at least one c between -3 and 5~~

(C) $f(c) = 0$ for at least one c between -3 and 5

(D) $f(c) = 0$ for at least one c between -2 and 7

27. Let g be a function defined by $g(x) = \begin{cases} \dfrac{\pi \sin x}{x}, & \text{if } x < 0 \\ a - bx, & \text{if } 0 \le x < 1 \\ \arctan x, & \text{if } x \ge 1 \end{cases}$

If g is continuous for all real numbers x, what are the values of a and b?

$\lim_{x\to 0^-} = \lim_{x \to 0^+} \rightarrow \frac{\pi \sin 0}{0} = a - b(0)$

L'Hopitals Rule

$\frac{\pi \cos 0}{1} = a - b(0)$

$\boxed{\pi = a}$

$\lim_{x\to 1^-} = \lim_{x\to 1^+} \rightarrow a - b(1) = \arctan(1)$

$a - b = \frac{\pi}{4}$

$\pi - b = \frac{\pi}{4}$

$-b = \frac{\pi}{4} - \frac{4\pi}{4}$

$\boxed{b = \frac{3\pi}{4}}$

28. Evaluate $\lim_{a\to 0} \frac{-1+\sqrt{1+a}}{a} \left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right) = \lim_{a\to 0} \frac{1-(1+a)}{a(-1-\sqrt{1+a})}$

$\lim_{a\to 0} \frac{-\cancel{a}}{\cancel{a}(-1-\sqrt{1+a})}$

$\frac{-1}{-1-\sqrt{1}} = \frac{-1}{-2} = \boxed{\frac{1}{2}}$

29. What is the value of a, if $\lim_{x\to 0} \frac{\sqrt{ax+9}-3}{x} = 1$?

$\frac{(ax+9)^{1/2}-3}{x}$

L'Hopitals Rule

$\lim_{x\to 0} \frac{\frac{1}{2}(ax+9)^{-1/2}(a)}{1}$

$= \frac{1}{2}(9)^{-1/2}(a)$

$= \frac{a}{2\sqrt{9}} = 1$

$\boxed{a = 6}$

30. $\lim_{x\to\infty} \frac{3+2x^2-x^4}{3x^4-5} =$ $-1/3$ (horizontal asymptote)

(A) -2 (B) $-\frac{1}{3}$ (C) $\frac{1}{5}$ (D) 1

31. What is $\lim_{x\to -\infty} \frac{x^3+x-8}{2x^3+3x-1} =$ $\frac{1}{2}$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) 2

horizontal asymptote → look @ the degree

Horizontal Asymp at $y=1$

32. Which of the following lines is an asymptote of the graph of $f(x)=\frac{x^2+5x+6}{x^2-x-12}$? $\frac{(\cancel{x+3})(x+2)}{(x-4)(\cancel{x+3})}$

Vertical Asympt at $x=4$

I. $x=-3$ → hole

II. $x=4$ V.A

III. $y=1$ H.A

(A) II only (B) III only (C) II and III only (D) I, II, anc

33. If the horizontal line $y=1$ is an asymptote for the graph of the function f, which of the follo statements must be true?

(A) $\lim_{x\to\infty} f(x)=1$ ← Horizontal Asymptote

(B) $\lim_{x\to 1} f(x)=\infty$

(C) $f(1)$ is undefined

(D) $f(x)=1$ for all x

34. If $x=1$ is the vertical asymptote and $y=-3$ is the horizontal asymptote for the graph of the function f, which of the following could be the equation of the curve?

(A) $f(x)=\frac{-3x^2}{x-1}$

$\frac{-3}{(x-1)}$

(B) $f(x)=\frac{-3(x-1)}{x+3}$

(C) $f(x)=\frac{-3(x^2-1)}{x-1}$

$\frac{-3x^2}{x^2}$ H.A $y=-3$

(D) $f(x)=\frac{-3(x^2-1)}{(x-1)^2}$ $\frac{-3x^2+3x}{(x-1)^2}$ = ← V.A $x=1$

35. What are all horizontal asymptotes of the graph of $y=\frac{6+3e^x}{3-3e^x}$ in the xy-plane?

(A) $y=-1$ only

(B) $y=2$ only

(C) $y=-1$ and $y=2$

(D) $y=0$ and $y=2$

Special Case: $\boxed{\text{b/c } e^x}$

$y=\frac{3}{-3} \rightarrow y=-1$

& $y=\frac{6}{3} \rightarrow y=2$