DIFFERENTIAL EQUATIONS REVIEW

NO CALCULATOR

$(-1)^2 + (-1) = 0$
$(1)^2 + (-5)^2 = 26$

critical pt   not critical pt

1. The points $(-1, -1)$ and $(1, -5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2 + y$. Which of the following must be true?

2nd Deriv. Test

$\frac{d^2y}{dx^2} = 2x + \frac{dy}{dx} = 2x + x^2 + y$

$(-1,-1) \rightarrow 2(-1) + (-1)^2 + (-1) = -2 < 0$

$f(x)$ is conc ↓ rel max

(A) $(1, -5)$ is a local maximum of f.

(B) $(1, -5)$ is a point of inflection of the graph of f.

(C) $(-1, -1)$ is a local maximum of f.

(D) $(-1, -1)$ is a local minimum of f.

(E) $(-1, -1)$ is a point of inflection of the graph of f.

2. Let k be a positive constant. Which of the following is a $\boxed{\text{logistic differential equation?}}$

(A) $\frac{dy}{dt} = kt$ (B) $\frac{dy}{dt} = ky$ (C) $\frac{dy}{dt} = kt(1-t)$ (D) $\frac{dy}{dt} = ky(1-t)$ (E) $\frac{dy}{dt} = ky(1-y)$

3. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - y$ with initial condition $f(1) = 3$. What is the approximation for $f(2)$ obtained by using $\boxed{\text{Euler's method}}$ with two steps of equal length starting at $x = 1$?

x y

(A) $-\frac{5}{4}$ (B) 1   (C) $\frac{7}{4}$ (D) 2   (E) $\frac{21}{4}$

x	y	dy/dx	dy
1	3	-2	(-2)(.5) = -1
1.5	2	-.5	-.5(.5) = -.25
2	1.75		

4. If $P(t)$ is the size of a population at time t, which of the following differential equations describes linear growth in the size of the population?

(A) $\frac{dP}{dt} = 200$ (B) $\frac{dP}{dt} = 200t$ (C) $\frac{dP}{dt} = 100t^2$ (D) $\frac{dP}{dt} = 200P$ (E) $\frac{dP}{dt} = 100P^2$

$P = 200t$
Linear

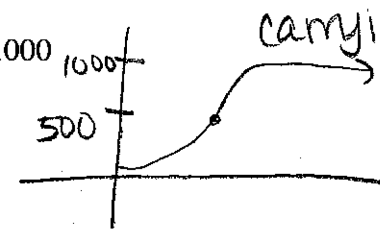
Logistic Diff eqn

5. A population y changes at a rate modeled by the differential equation $\frac{dy}{dt} = 0.2y(1000 - y)$, where t is measured in years. What are all values of y for which the population is $\boxed{\text{increasing at a decreasing rate?}}$

(A) 500 only   (B) $0 < y < 500$ only   (C) $500 < y < 1000$ only

(D) $0 < y < 1000$ (E) $y > 1000$

carrying capacity = 100



incr & concave ↓

6. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 2x + y$ with initial condition $f(1) = 0$. What is the approximation for $f(2)$ obtained by using Euler's method with two steps of equal length, starting at $x = 1$?

X Y

(A) 0 (B) 1 (C) 2.75 (D) 3 (E) 6

x	y	dy/dx	dy
1	0	2	2(.5) = 1
1.5	1	4	4(.5) = 2
2	3		

7. For $0 < P < 100$, which of the following is an antiderivative of $\frac{1}{100P - P^2}$?

$\frac{1}{P(100-P)} = \frac{A}{P}\,(100-P) + \frac{B}{100-P}\,(P)$

(A) $\frac{1}{100}\ln(P) - \frac{1}{100}\ln(100-P)$

(B) $\frac{1}{100}\ln(P) + \frac{1}{100}\ln(100-P)$

$1 = A(100-P) + B(P)$

$P = 100$: $1 = B(100)$, $B = 1/100$

(C) $100\ln(P) - 100\ln(100-P)$

(D) $\ln(100P - P^2)$

(E) $\frac{1}{50P^2 - \frac{P^3}{3}}$

$P = 0$: $1 = A(100)$, $A = 1/100$

$\int \frac{1/100\, dP}{P} + \int \frac{1/100}{100-P}\, dP$

$u = 100 - P$
$du = -1$

$\frac{1}{100}\ln|P| - \frac{1}{100}\ln|100-P|$

8. Which of the following is the solution to the differential equation $\frac{dy}{dx} = e^{y+x}$ with initial condition $y(0) = -\ln 4$?

(A) $y = -x - \ln 4$

(B) $y = x - \ln 4$

(C) $y = -\ln\left(-e^x + 5\right)$

(D) $y = -\ln\left(-e^x + 3\right)$

(E) $y = \ln\left(-e^x + 3\right)$

$\frac{dy}{dx} = e^y \cdot e^x$

$\frac{dy}{e^y} = e^x\, dx$

$\int e^{-y}\, dy = \int e^x\, dx$

$-e^{-y} = e^x + C$

$y(0) = -\ln 4$

$-e^{-(-\ln 4)} = e^0 + C$

$-e^{\ln 4} = 1 + C$

$-4 = 1 + C$

$C = -5$

$\ln -e^{-y} = e^x - 5$ ln

$y = \ln(e^x - 5)$

$y = -\ln(-e^x + 5)$

9. What is the particular solution to the differential equation $\frac{dy}{dx} = xy^2$ with initial condition $y(2) = 1$?

(A) $y = e^{\frac{x^2}{2} - 2}$

(B) $y = e^{\frac{x^2}{2}}$

(C) $y = -\frac{2}{x^2}$

(D) $y = \frac{2}{6 - x^2}$

(E) $y = \frac{6 - x^2}{2}$

$\int \frac{dy}{y^2} = \int x\, dx$

$-\frac{1}{y} = \frac{1}{2}x^2 + C$

$y(2) = 1$ $\quad -\frac{1}{1} = \frac{1}{2}(2)^2 + C$

$-1 = 2 + C \quad C = -3$

$\left(-\frac{1}{y} = \frac{1}{2}x^2 - 3\right) y$

$-1 = \left(\frac{1}{2}x^2 - 3\right) y$

$\frac{-1}{\frac{1}{2}x^2 - 3} = y$

$\frac{-2}{x^2 - 6} = y \quad y = \frac{2}{6 - x^2}$

$x^2 - 7x + 10 = (x-2)(x-5)$

$\int dy = \int \frac{1}{x^2-7x+10}\,dx$

10. What is the general solution to the differential equation $\frac{dy}{dx} = \frac{1}{x^2-7x+10}$?

(A) $\ln|(x-2)(x-5)| + C$

(B) $\frac{1}{3}\ln|(x-2)(x-5)| + C$

(C) $\frac{1}{3}\ln\left|\frac{2x-7}{(x-2)(x-5)}\right| + C$

(D) $\frac{1}{3}\ln\left|\frac{x-2}{x-5}\right| + C$

(E) $\frac{1}{3}\ln\left|\frac{x-5}{x-2}\right| + C$

$y = \int \frac{-1/3}{x-2}\,dx + \int \frac{1/3}{x-5}\,dx$

$y = -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$

$y = \frac{1}{3}\ln\left|\frac{x-5}{x-2}\right|$

$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$

$1 = A(x-5) + B(x-2)$

$x=5: \quad 1 = B(3) \quad B = 1/3$

$x=2: \quad 1 = A(-3) \quad A = -1/3$

$\frac{d}{dx} x^{1/2} = \frac{1}{2}x^{-1/2}$

11. For $x > 0$, $\frac{d}{dx}\left(\int_1^{\sqrt{x}} \frac{1}{1+t^2}\,dt\right) =$

$\frac{1}{1+(\sqrt{x})^2} = \frac{1}{1+x^2} \cdot \frac{d}{dx} = \frac{1}{1+x^2}\left(\frac{1}{2\sqrt{x}}\right)$

(A) $\frac{1}{2\sqrt{x}(1+x)}$ (B) $\frac{1}{2\sqrt{x}(1+\sqrt{x})}$ (C) $\frac{1}{1+x}$ (D) $\frac{\sqrt{x}}{1+x}$ (E) $\frac{1}{1+\sqrt{x}}$

12. The number of antibodies y in a patient's bloodstream at time t in increasing according to a logistic differential equation. Which of the following could be the differential equation?

(A) $\frac{dy}{dt} = 0.025t$ (B) $\frac{dy}{dt} = 0.025t(5000-t)$ (C) $\frac{dy}{dt} = 0.025y$

(D) $\frac{dy}{dt} = 0.025(5000-y)$ (E) $\frac{dy}{dt} = 0.025y(5000-y)$

13. (calculator active)

initial

At time $t = 0$ years, a forest preserve has a population of 1500 deer. If the rate of growth of the population is modeled by $R(t) = 2000e^{0.23t}$ deer per year, what is the population at time $t = 3$?

(A) 3987 (B) 5487 (C) 8641 (D) 10,141 (E) 12,628

$\int_0^3 2000e^{.23t} = 8641 + 1500 = 10,141$

✱ 14.

x	$f'(x)$
1	0.2
1.5	0.5
2	0.9

The table gives values of $f'(x)$, the derivative of a function f. If $f(1) = 4$, what is the approximation to $f(2)$ obtained by using Euler's method with a step size of 0.5?

x y

(A) 2.35 (B) 3.64 (C) 4.35 (D) 4.70 (E) 4.80

x	y	dy/dx	dy
1	4	0.2	0.2(.5) = .1
1.5	4.1	0.5	0.5(0.5) = 0.25
2	4.35		

FREE RESPONSE

15.

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

$\frac{dP}{dt} = \frac{1}{60}P(12 - P)$ carrying capacity = 12

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(a) $P(0) = 3$, $\lim_{t\to\infty} P(t) = \boxed{12}$

$P(0) = 20$, $\lim_{t\to\infty} P(t) = \boxed{12}$

carrying capacity

(b) population grows the fastest when carrying capacity/2 $\frac{12}{2} = \boxed{6}$

ⓒ $\frac{dy}{dt} = \frac{y}{5}\left(1 - \frac{t}{12}\right)$

$\int \frac{dy}{y} = \int \frac{1}{5}\left(1 - \frac{t}{12}\right) dt$

$\ln y = \int \frac{1}{5} - \frac{1}{60} t \, dt$

$\ln y = \frac{1}{5} t - \frac{1}{120} t^2 + C$

$y(0) = 3$

$\ln 3 = \frac{1}{5}(0) - \frac{1}{120}\cancel{(0)^2} + C$

$C = \ln 3$

$\cancel{e}^{\ln y} = e^{\frac{1}{5} t - \frac{1}{120} t^2 + \ln 3}$

$y = e^{\left(\frac{1}{5} t - \frac{1}{120} t^2 + \ln 3\right)}$

$\boxed{y = 3e^{\left(\frac{1}{5} t - \frac{1}{120} t^2\right)}}$