Convergence of Series

TEST	SERIES	CONDITION(S) OF CONVERGENCE	CONDITION(S) OF DIVERGENCE	COMMENTS
***nth* Term**	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence!
Geometric Series	$\sum_{n=0}^{\infty} a r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
***p*-Series**	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$\lim_{n\to\infty} a_n = 0$ $0 < a_{n+1} \leq a_n$		Remainder: $\lvert R\rvert \leq a_{n+1}$
Integral (f is continuous, positive, decreasing)	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\, dx = L$	$\int_1^{\infty} f(x)\, dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\, dx$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = 1$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$
Direct Comparison $a_n > 0$ and $b_n > 0$	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $a_n > 0$ and $b_n > 0$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Examples ** Determine if series converges or diverges.

① $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n)}$ $\boxed{\text{converges Telescoping}}$

② $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$ $\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$ $\boxed{\text{diverges nth term test}}$

③ $\sum_{n=1}^{\infty} \frac{n^5}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\boxed{\text{p-series test converges}}$

④ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ $\frac{1}{n+1} \le \frac{1}{n}$ ✓ $\boxed{\text{converges by Alt. Series Test}}$

⑤ $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\lim_{n\to\infty} \left| \frac{2^{n+1}}{\underset{(n+1)(n!)}{\cancel{(n+1)!}}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = \frac{2}{\infty} = 0 < 1$

$\boxed{\text{Converges ratio test}}$

⑥ $\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n$ $\boxed{\text{Converges Geometric Series}}$

Answer Key

CONVERGENCE OF SERIES REVIEW

NO CALCULATOR

1. Which of the following series converge? $\lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{100}}\cdot\frac{n^{100}}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)n^{100}}{(n+1)^{100}}\right| = \lim_{n\to\infty}\left|\frac{n^{101}}{n^{100}}\right|$ $\frac{\text{big}}{\text{small}}$ = div

I. $\sum_{n=1}^{\infty}\frac{8^n}{n!}$ ~~II.~~ $\sum_{n=1}^{\infty}\frac{n!}{n^{100}}$ III. $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

$\lim_{n\to\infty}\left|\frac{8^{n+1}}{(n+1)!}\cdot\frac{n!}{8^n}\right| = \lim_{n\to\infty}\left|\frac{8}{n+1}\right| = \frac{8}{\infty} = 0 < 1$

converges by ratio test

Telescoping Series → converges

2. What is the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(x-4)^{2n}}{3^n}$? Ratio Test $\lim_{n\to\infty}\left|\frac{(x-4)^{2n+2}}{3^{n+1}}\cdot\frac{3^n}{(x-4)^{2n}}\right|$

(A) $2\sqrt{3}$ (B) 3 (C) $\sqrt{3}$ (D) $\frac{\sqrt{3}}{2}$ (E) 0 $= \lim_{n\to\infty}\left|\frac{(x-4)^2}{3}\right| = \frac{(x-4)^2}{3}$

$\frac{(x-4)^2}{3} < 1 \rightarrow (x-4)^2 < 3$

$x-4 < \sqrt{3}$ $R=\sqrt{3}$

3. The power series $\sum_{n=0}^{\infty}a_n(x-3)^n$ converges at $x=5$. Which of the following must be true?

~~(A)~~ The series diverges at $x = 0$.

~~(B)~~ The series diverges at $x = 1$.

~~(C)~~ The series converges at $x = 1$.

(D) The series converges at $x = 2$.

~~(E)~~ The series converges at $x = 6$.

center = 3

could be either → 1 (2) ✓ 3 (5) converges 6

Radius = at least 2

4. For what values of p will both series $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ and $\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n$ converge?

(A) $-2 < p < 2$ only

(B) $-\frac{1}{2} < p < \frac{1}{2}$ only

(C) $\frac{1}{2} < p < 2$ only

(D) $p < \frac{1}{2}$ and $p > 2$

(E) There are no such values of p.

$\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ p-series test

$\frac{1}{n^{2(1)}} = \frac{1}{n^2}$ converges if p=1

$\frac{1}{n^{2(\frac{1}{2})}} = \frac{1}{n}$ diverges if p=1/2

$\therefore p > \frac{1}{2}$

$\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n$ Geom. Test

$\left(\frac{2}{2}\right)^n = (1)^n$ diverge

$\therefore p < 2$

5. If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following must be true?

Ratio Test converges

(A) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ <1 ✓

(B) $|a_n| < 1$ for all n

(C) $\sum_{n=1}^{\infty} a_n = 0$

(D) $\sum_{n=1}^{\infty} na_n$ diverges.

(E) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

6. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ converges?

(A) $-\frac{5}{2} < x < -\frac{1}{2}$

(B) $-\frac{5}{2} < x \le -\frac{1}{2}$

(C) $-\frac{5}{2} \le x < -\frac{1}{2}$

(D) $-\frac{1}{2} < x < \frac{1}{2}$

(E) $x \le -\frac{1}{2}$

$\left(-\frac{5}{2}, \frac{1}{2}\right]$

$x=-\frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(-\frac{5}{2}+\frac{3}{2}\right)^n = \sum_1 \frac{(-1)^n}{n}(-1)^n$
$= \sum_1 \frac{1}{n}$ p-series diverges

$x=-\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(-\frac{1}{2}+\frac{3}{n}\right)^n$
$= \sum_1 \frac{(-1)^n}{n}(1)^n = \sum_1 \frac{(-1)^n}{n}$ Alt. Series Test $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ $\frac{1}{n+1} \le \frac{1}{n}$ ✓ Conv.

7. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \ge 1$. If $\sum_{n=1}^{\infty} a_n$ converges, which of the following must be true?

Direct Comparison Test

(A) If $a_n \le b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(B) If $a_n \le b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(C) If $b_n \le a_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(D) If $b_n \le a_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(E) If $b_n \le a_n$, then the behavior of $\sum_{n=1}^{\infty} b_n$ cannot be determined from the information given.

a_n b_n
bigger
$\sum a_n$ converges, then $\sum_1 b_n$ conv.

8. Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n}$?

$n=2$: $\frac{(-1)^2}{\sqrt{2}+(-1)^2} = \frac{1}{\sqrt{2}+1}$ $n=3$: $\frac{(-1)^3}{\sqrt{3}+(-1)^3} = \frac{-1}{\sqrt{3}-1}$ $n=4$: $\frac{(-1)^4}{\sqrt{4}+1}$

I. The series is alternating. True $\rightarrow (-1)^n$

II. $|a_{n+1}| \le |a_n|$ for all $n \ge 2$ False, denom. adds 1 then subtracts 1

III. $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0$

(A) None

(B) I only

(C) I and II only

(D) I and III only

(E) I, II, and III

9. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$?

Ratio Test

$\lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-3)^n}\right|$

$\lim_{n\to\infty} \left|\frac{(x-3)(n)}{(n+1)(2)}\right| = \frac{(x-3)}{2} < 1$

$(x-3) < 2$

$R=2$

(A) $1 < x < 5$

(B) $1 \le x < 5$

(C) $1 \le x \le 5$

(D) $2 < x < 4$

(E) $2 \le x \le 4$

$x=1$: $\sum_1 \frac{(-2)^n}{n \cdot 2^n} = \sum_1 \frac{(-1)^n}{n}$ alt. series conv.

$x=5$: $\sum_1 \frac{(2^n)}{n2^n}$ $\sum_1 \frac{1}{n}$ p-series div

$[1 \quad 3 \quad 5)$