2006 AP® CALCULUS AB
Question 6

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2,\ f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

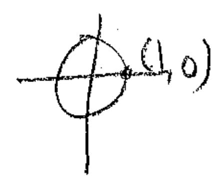
(a)

$$g(x) = e^{ax} + f(x)$$
$$g'(x) = (e^{ax})(a) + f'(x)$$
$$g'(0) = e^{a(0)}(a) + f'(0)$$
$$= e^{0}(a) + -4$$
$$= \boxed{a-4}$$

$$g'(x) = ae^{ax} + f'(x)$$
$$g''(x) = ae^{ax}(a) + f''(x)$$
$$g''(0) = ae^{a(0)}(a) + f''(0)$$
$$= ae^{0}(a) + 3$$
$$= \boxed{a^2+3}$$

(b)

$$h(x) = \cos(kx)f(x)$$
$$h'(x) = -\sin(kx)\cdot(k)f(x) + \cos(kx)f'(x)$$
$$h'(0) = -\sin(k(0))(k)f(0) + \cos(k(0))f'(0)$$
$$= (-\sin 0)(k)(2) + (\cos 0)(-4)$$
$$= \cancel{(-0)(k)(2)} + (1)(-4)$$
$$= -4 \text{ slope of tangent}$$



$$h(0) = \cos(0k)f(x)$$
$$\cos(0)f(0)$$
$$(1)(2)$$

$(0,2) \leftarrow$ point of tangency

$$\boxed{y-2=-4x}$$

AP Calculus
Unit 3 – Rules of Differentiation

$f(3)=5 \quad f^{-1}(5)=3$

Day 6 Notes: Derivatives of Inverse Functions

Given a function, $f(x)$, the inverse function, $f^{-1}(x)$, is numerically defined to be

the function that results from switching the domain & range of $f(x)$.

Graphical Representation of the Inverse	Analytical Representation of the Inverse
$f^{-1}(x)$ is reflection of $f(x)$ over $y=x$.	$f(f^{-1}(x))=x$ $f^{-1}(f(x))=x$

Consider the two functions, $f(x)$ and $g(x)$, represented numerically below. Answer the questions that follow.

x	$f(x)$	$g(x)$
-2	3	1
1	2	-2

Complete the table of values below.

x	$f^{-1}(x)$
3	-2
2	1

Complete the table of values below.

x	$g^{-1}(x)$
1	-2
-2	1

Find the value of $f\left(g^{-1}(-2)\right)$.

y

$f(1) = \boxed{2}$

Find the value of $f^{-1}(f(1))$.

$f^{-1}(2) = \boxed{1}$

Find the value of $g^{-1}\left(f^{-1}(2)\right)$

$g^{-1}(1) = \boxed{-2}$

Find the value of $f^{-1}\left(g\left(g^{-1}(1)\right)\right)$.

$f^{-1}(g(-2))$

$f^{-1}(1) = \boxed{\text{undefined}}$

Finding a Formula for the Derivative of an Inverse

Differentiate both sides of the equation below.

Chain Rule

$$f[f^{-1}(x)]=x$$

$$f'[f^{-1}(x)] \cdot [f^{-1}(x)]' = 1$$

$$\boxed{[f^{-1}(x)]' = \frac{1}{f'[f^{-1}(x)]}}$$

Suppose that $f(x) = 3x + 2$ and $f'(-2) = 3$. What is the value of $[f^{-1}(-4)]'$?

$$[f^{-1}(-4)]' = \frac{1}{f'[f^{-1}(-4)]} = \frac{1}{f'(-2)} = \boxed{\frac{1}{3}}$$

$y = 3x + 2$
$-4 = 3x + 2$
$-6 = 3x$
$x = -2$

Given to the right is a table of values for f, g, f', and g'. Use the values in the table to find each indicated value in the boxes below.

x	f	g	f'	g'
−2	1	2	0	3
0	−4	−3	−1	2
1	3	−2	2	1
3	1	1	−3	−2

Find the value of $[f^{-1}(3)]'$. $= \frac{1}{f'[f^{-1}(3)]}$

$$= \frac{1}{f'[1]} = \boxed{\frac{1}{2}}$$

Find the value of $[g^{-1}(-2)]'$. $= \frac{1}{g'[g^{-1}(-2)]}$

$$= \frac{1}{g'(1)} = \frac{1}{1} = \boxed{1}$$

Find the value of $[g^{-1}(1)]'$. Then, find the equation of the line tangent to the graph of g^{-1} when $x = 1$.

$$\frac{1}{g'[g^{-1}(1)]} = \frac{1}{g'(3)} = \boxed{\frac{1}{-2}}$$

slope of tangent

$g^{-1}(1) = 3$ (1,3) point of tangent

$$\boxed{y - 3 = -\frac{1}{2}(x-1)}$$

Estimate the value of $f'(2)$. Based on this value, what conclusion can be reached about the graph of f when $x = 2$? Explain your reasoning.

(1, 3) & (3, 1)

$$f'(2) \approx \frac{1-3}{3-1} = \frac{-2}{2} \approx \boxed{-1}$$

Since $f'(2) < 0$, then $f(x)$ is decreasing at $x = 2$.