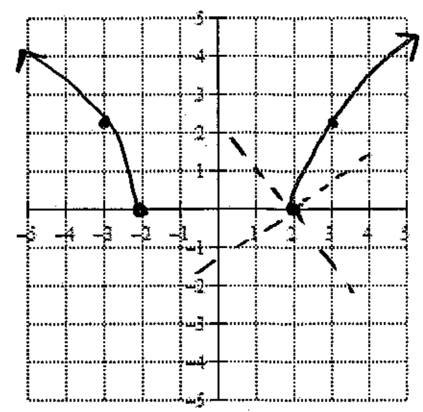
AP Calculus
Unit 3 – Rules of Differentiation

Day 5 Notes: The Relationship Between Continuity & Differentiability

Example 1:

Consider the function $f(x) = \sqrt{x^2 - 4}$ at $x = 2$.

x	y
-2	0
2	0
-3	2.24
3	2.24



Based on the graph, if $f(x)$ continuous at $x = 2$? Explain your reasoning.

① $f(2) = 0$, $f(2)$ is defined ✓

② $\lim_{x\to 2^-} f(x) =$ D.N.E, so $\lim_{x\to 2} f(x)$ D.N.E.

$\therefore$ $f(x)$ is <u>NOT</u> continuous at $x=2$.

Find the value of $f'(2)$ to determine if $f(x)$ is differentiable at $x = 2$.

$f(x) = (x^2-4)^{1/2}$

$f'(x) = \cancel{\frac{1}{2}}(x^2-4)^{-1/2}(\cancel{2}x)$

$f'(x) = \dfrac{x}{\sqrt{x^2-4}} \rightarrow f'(2) = \dfrac{2}{\sqrt{2^2-4}} = \dfrac{2}{0} =$ undefined

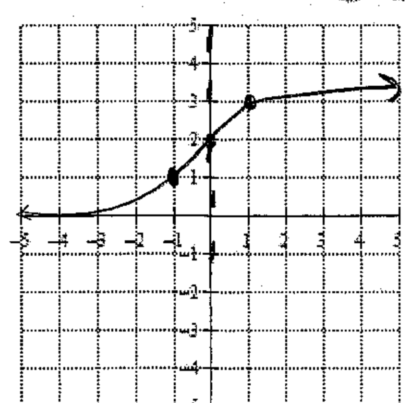
f(x) is not differentiable at x=2 b/c f'(2) is undefined.

Graphically, f(x) is not diff. b/c more than 1 tangent line can be drawn to f(x) at x = 2.

Example 2:

Consider the function $f(x) = x^{\frac{1}{3}} + 2$ at $x = 0$.

x	y
-1	1
0	2
1	3



Based on the graph, if $f(x)$ continuous at $x = 0$? Explain your reasoning.

① $f(0) = 2$, $f(0)$ is defined ✓

② $\lim_{x\to 0^-} f(x) = 2$ & $\lim_{x\to 0^+} f(x) = 2$, so $\lim_{x\to 0} f(x)$ exists ✓

③ $f(0) = 2 = \lim_{x\to 2} f(x)$ ✓

$\therefore$ $f(x)$ is continuous at $x=0$.

Find the value of $f'(0)$ to determine if $f(x)$ is differentiable at $x = 0$.

$f(x) = x^{1/3} + 2$

$f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$

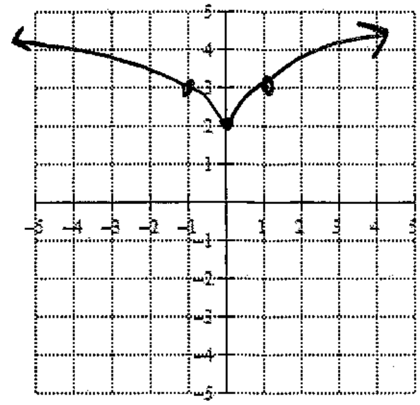
$f'(0) = \dfrac{1}{3\sqrt[3]{0^2}} = \dfrac{1}{0} =$ undefined $\rightarrow$ $\boxed{\text{f(x) is not differentiable at } x=0}$

Graphically, f(x) is not diff. b/c x=0 is a point of inflection. (curve changes direction)

Example 3:

Consider the function $f(x) = x^{\frac{2}{3}} + 2$ at $x = 0$.

x	y
-1	3
0	2
1	3



Based on the graph, if $f(x)$ continuous at $x = 0$? Explain your reasoning.

① $f(0) = 2$, $f(0)$ is defined ✓

② $\lim_{x\to 0^-} f(x) = 2$, $\lim_{x\to 0^+} f(x) = 2$, so $\lim_{x\to 0} f(x)$ exists ✓

③ $f(0) = \lim_{x\to 0} f(x) = 2$ $\therefore f(x)$ is cont. at $x=0$.

Find the value of $f'(0)$ to determine if $f(x)$ is differentiable at $x = 0$.

$f'(x) = \frac{2}{3}x^{-1/3} + 0$

$f'(x) = \frac{2}{3\sqrt[3]{x}}$ $f'(0) = \frac{2}{3\sqrt[3]{0}} = \frac{2}{0}$ = undefined

$\boxed{f(x) \text{ is not. diff. at } x=0.}$

Graphically, f(x) is not diff. b/c f(x) has a cusp (graph comes to sharp point) at x=0.

Graphically, a function is $\boxed{\text{not}}$ differentiable if....

1) the function is NOT continuous

2) if there is a point of inflection

3) if a cusp exists.

In order for a function to be differentiable at a value of x, then two things must be true:

1) f(x) must be continuous at x=a.

2) $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$ which means $\lim_{x\to a} f'(x)$ must exist.

Example 4:

$(x+1)^{1/2}$

Consider the function $g(x)=\begin{cases}\sqrt{x+1}, & 0\le x\le 3\\ 5-x, & 3<x\le 5\end{cases}$ to answer the following questions.

Is $g(x)$ continuous at $x=3$? Show the complete analysis.

① $g(3)=\sqrt{3+1}=2$, $g(3)$ is defined ✓

② $\lim_{x\to 3^-} g(x)=\sqrt{3+1}=2$ & $\lim_{x\to 3^+} g(x)=5-3=2$, so $\lim_{x\to 3} g(x)$ exists ✓

③ $g(3)=\lim_{x\to 3} g(x)=2$ ✓

$\therefore g(x)$ is continuous at $x=3$

Is $g(x)$ differentiable at $x=3$? Show the complete analysis.

① As shown above, $g(x)$ is continuous at $x=3$.

② $g'(x)=\begin{cases}\frac{1}{2}(x+1)^{-1/2}(1), & 0\le x\le 3\\ -1, & 3<x\le 5\end{cases} \rightarrow g'(x)=\begin{cases}\frac{1}{2\sqrt{x+1}}, & 0\le x\le 3\\ -1, & 3<x\le 5\end{cases}$

$\lim_{x\to 3^-} g'(x)=\frac{1}{2\sqrt{3+1}}=\frac{1}{4}$, $\lim_{x\to 3^+} g'(x)=-1$

$\lim_{x\to 3^-} g'(x)\neq\lim_{x\to 3^+} g'(x)$ so $g(x)$ is not differentiable at $x=3$

Example 5:

For what values of k and m will the function below be both continuous and differentiable at $x=3$?

$k(x+1)^{1/2}$

$h(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases} \rightarrow h'(x)=\begin{cases}\frac{1}{2}k(x+1)^{-1/2}(1)\\ m\end{cases}$

$h(3)$ is defined ✓

For $h(x)$ to be continuous at $x=3$

$\lim_{x\to 3^-} h(x)=\lim_{x\to 3^+} h(x)$.

$\downarrow \qquad \downarrow$

$k\sqrt{3+1}=m(3)+2$

$2k=3m+2$

For $h(x)$ to be differentiable at $x=3$,

$\lim_{x\to 3^-} h'(x)=\lim_{x\to 3^+} h'(x)$

$\downarrow$

$\frac{1}{2}k(3+1)^{-1/2}=m$

$\frac{k}{2\sqrt{4}}=m$

$\frac{k}{4}=m$

$k=4m$

$2k=3m+2$

$2(4m)=3m+2$

$8m=3m+2$

$5m=2$

$\boxed{m=2/5}$

$k=4m$

$k=4(2/5)$ $\boxed{k=8/5}$